PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 1, 1971

STRUCTURE OF SEMIPRIME (p, q) RADICALS

T. L. GOULDING AND AUGUSTO H. ORTIZ

In this note, the structure of the semiprime (p, q) radicals is investigated. Let $p(x)$ and $q(x)$ be polynomials over the integers. An element a of an arbitrary associative ring R is called (p, q)-*regular* if $a \in p(a) \cdot R \cdot q(a)$. A ring R is (p, q)-*regular* if every element of R is (p, q)-regular. It is easy to prove that (p, q)-regularity is a radical property and also that it is a semiprime radical property (meaning that the radical of a ring is a semiprime ideal of the ring) if and only if the constant coefficients of $p(x)$ and $q(x)$ are ± 1. It is shown that every (p, q)-semisimple ring is isomorphic to a subdirect sum of rings which are either right primitive or left primitive.

Our results follow the ideas in [1]. However, a direct application of the results of [1] is not possible here because condition P_1 [1, p. 302] is not always satisfied in the present case.

Let R be an arbitrary associative ring. Let $p(x) = 1 + n_1 x + \cdots + n_k x^k$ be a polynomial over the integers. For each element $a \in R$, let $F_R(a) = p(a) \cdot R$. In what follows we take $q(x) = 1$. Thus an element a of R is called $(p, 1)$-*regular* if $a \in F_R(a)$. A ring R is called $(p, 1)$-*regular* if every element in R is $(p, 1)$-regular. We shall denote the $(p, 1)$ radical property by F.

A right ideal I of R will be called $(p, 1)$-*modular* if there exists an element $e \notin I$ such that $F_R(e) + eI \subset I$. In order to specify the element e we shall sometimes say that I is $(p, 1)_e$-*modular*. An ideal P of R will be called $(p, 1)$-*primitive* if P is the largest two sided ideal contained in some maximal $(p, 1)_e$-modular right ideal for some e. For a right ideal M of R, let $(M: R) = \{a \in R \mid Ra \subset M\}$ and let $p_0(x) = p(x) - 1$ throughout this paper.

LEMMA 1. *An ideal P of R is $(p, 1)$-primitive if and only if there exists $e \in R$ and a maximal $(p, 1)_e$-modular right ideal M such that $P = (M: R)$.*

Proof. It is clear that $(M: R)$ is a two sided ideal of R. Moreover if $a \in (M: R)$, then $a = p(e) \cdot a - p_0(e) \cdot a \in F_R(e) + Ra \subset M$. Finally if K is an ideal contained in M, then $RK \subset K \subset M$. Hence $K \subset (M: R)$. Thus $(M: R)$ is the largest two sided ideal contained in M.

LEMMA 2. *If I is a $(p, 1)_e$-modular right ideal of R and if $b \in I$, then*

$$F_R(e + b) \subset I.$$