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WEIGHTED LATTICE PATHS

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A lattice path in the plane from $(0, 0)$ to (m, n) with weighted horizontal, vertical, and diagonal steps will be called a weighted lattice path. We determine the number of unrestricted weighted lattice paths, the number of paths below a line, and the number of paths which must remain between two parallel lines with unit slope. We also obtain generating functions for the number of paths which remain below the line $y = x$; these extend results obtained by Carlitz and Riordan for the ballot numbers.

The number, $\binom{m+n}{n}$, of unrestricted minimal lattice paths from $(0, 0)$ to (m, n) is an old and well-known result, see MacMahon [5, I, p. 167] or Feller [3, p. 68]. The Catalan number $c_n = \binom{2n}{n}/(n+1)$ is the number of paths from $(0, 0)$ to (n, n) which never rise above the line $y = x$ (c.f. [1]). The enumeration of the paths which must remain below the line $y = ax + b$ has been obtained by Lyness [4], Mohanty and Narayana [7], and Carlitz, Roselle, and Scoville [2]. Recently several authors [6], [8], and [10] have considered the same problem when diagonal steps are allowed in addition to the usual horizontal and vertical steps, and Stanton and Cowan [11] studied the resulting numbers in a different connection.

In [6] Mohanty and Handa enumerate the unrestricted lattice paths from $(0, 0)$ to (m, n) where μ diagonal steps are allowed at each position. They also determine the number of such paths which must remain below the line $x = ay$, and they indicate that a formula can be obtained for the number of paths which must remain below $x = ay - b$, where a and b are nonnegative integers. All of these results have been extended to weighted lattice paths in the present paper.

We remark that we will treat the case of weighted lattice paths bounded above and below by functions which are not necessarily straight lines in a paper now under preparation. This paper will also contain a generalization of a problem posed by Elwyn Berlekamp and solved in [2].

2. Unrestricted lattice paths. By a weighted lattice path we mean a path from $(0, 0)$ to (n, k) where a lattice point (i, j) may be approached from any of the lattice points $(i, j-1)$, $(i-1, j)$, or $(i-1, j-1)$, and the horizontal steps are assigned the weight x, the