PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 1, 1971

RAMSEY BOUNDS FOR GRAPH PRODUCTS

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Here we show that Ramsey numbers $M(k_1, \cdots, k_n)$ give sharp upper bounds for the independence numbers of product graphs, in terms of the independence numbers of the factors.

The Ramsey number $M(k_1, \cdots, k_n)$ is the smallest integer m with the property that no matter how the $\binom{m}{2}$ edges of the complete graph on m nodes are partitioned into n colors, there will be at least one index i for which a complete subgraph on k_i nodes has all of its edges in the ith color. Ramsey's Theorem tells that these numbers exist but only a few exact values are known.

The complement graph $\bar{G}$ has the same nodes as G and the complementary set of edges.

The independence number $\alpha(G)$ of a graph G, is the largest number of nodes in any complete subgraph of $\bar{G}$.

The product $G_1 \times \cdots \times G_n$ of graphs $G_1, \cdots, G_n$ is the graph whose nodes are all the ordered n-tuples $(a_1, \cdots, a_n)$ in which a_i is a node of G_i for each i from 1 to n, and whose edges are as follows. A set of two nodes $\{(a_1, \cdots, a_n), (b_1, \cdots, b_n)\}$ will be an edge of $G_1 \times \cdots \times G_n$ if and only if the nodes are distinct and for each i from 1 to n, $a_i = b_i$ or $\{a_i, b_i\}$ is an edge of G_i.

THEOREM 1. *For arbitrary graphs $G_1, \cdots, G_n$*

$$\alpha(G_1 \times \cdots \times G_n) < M(\alpha(G_1) + 1, \cdots, \alpha(G_n) + 1) .$$

Proof. We have a complete subgraph of $\overline{G_1 \times \cdots \times G_n}$ on $\alpha(G_1 \times \cdots \times G_n)$ nodes. Its edges can be n colored by the following rule: give $\{(a_1, \cdots, a_n), (x_1, \cdots, x_n)\}$ color i if i is the first index for which $\{a_i, x_i\}$ is an edge of $\bar{G}_i$.

With this coloration any case where all the edges on k nodes have color i requires a complete k subgraph of $\bar{G}_i$ and so requires $k < \alpha(G_i) + 1$. With the definition of the Ramsey number this ensures that

$$\alpha(G_1 \times \cdots \times G_n) < M(\alpha(G_1) + 1, \cdots, \alpha(G_n) + 1) .$$

THEOREM 2. *If $k_1, \cdots, k_n$ are given, there exist graphs $G_1, \cdots, G_n$ such that for each index i from 1 to n, $\alpha(G_i) = k_i$ and*

$$\alpha(G_1 \times \cdots \times G_n) = M(k_1 + 1, \cdots, k_n + 1) - 1 .$$