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TRIANGULAR MATRICES WITH THE ISOCLINAL PROPERTY

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Consider the system V_n of $n \times n$, lower triangular matrices over the real numbers with the usual operations of addition, multiplication and scalar multiplication and with the additional property that $a_{i+1,j+1} = a_{i,j}$ (isoclinal). It is shown that V_n is a commutative vector algebra. The principal theorem (§ 3) establishes the existence of an algebraic mapping of V_n into a ring of rational functions. This mapping associates a special set of basis elements in V_n with the classically known Eulerian Polynomials.

Some properties of the space V_n are outlined in § 2. Section 4 gives an application of the main theorem to a problem which motivated this study, namely, the inversion of certain matrices in V_n for arbitrary dimension n. The matrices with first columns $[1^m, 2^m, \cdots, n^m]$, $m = 0, 1, 2, \cdots$, are considered in particular.

2. Properties.

2.1. *Nomenclature.* A matrix $A = \{a_{i,j}\}$ is called isoclinal if $a_{i+1,j+1} = a_{i,j}$ for all values of the indices permitted. Further we designate by V_n the class of $n \times n$ lower-triangular, isoclinal (L.T.I.) matrices (over the reals).

REMARK. The isoclinal property has appeared in studies of commutativity, under other names; for example see [4].

THEOREM 2.2. *The class V_n is a commutative sub-ring of matrices. Further, if $A \in V_n$ is nonsingular then $A^{-1} \in V_n$.*

Proof. A simple computation using the L.T.I. property will show multiplicative closure. Now, for $A, B \in V_n$ let $\{a_i\}$, $\{b_i\}$ be the elements of their first columns; these clearly define the matrices. The first column of AB is given by the Cauchy Product formula $\sum_{j=1}^{k}, a_j b_{k-j+1}$ for $k = 1, 2, \cdots, n$, which is commutative. Finally, if $A \in V_n$ is nonsingular then its diagonal element $a_1 \neq 0$ and the system $a_1 x_1 = 1$, $\sum_{j=1}^{k}, a_j x_{k-j+1} = 0$ is solvable. Hence $X \in V_n$ and $X = A^{-1}$.

The algebra of V_n is closely allied to that of the polynomials over the reals, $P(Y)$. Let $A \in V_n$ be given by its first column $\{a_i\}$. Define ϕ_n: $V_n \to P(Y)$ as the injection, $\phi_n(A) = \sum_{j=1}^{n}, a_j Y^{j-1}$ and let π_n: $P(Y) \to V_n$ be the projection. We then have: