PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 1, 1971

ON UNCONDITIONALLY CONVERGING SERIES AND BIORTHOGONAL SYSTEMS IN A BANACH SPACE

GREGORY F. BACHELIS AND HASKELL P. ROSENTHAL

Our main result is as follows: Let B be a Banach space containing no subspace isomorphic (linearly homeomorphic) to l_∞, and let $\{(b_n, \beta_n)\}$ be a biorthogonal sequence in B such that (β_n) is total. If $x \in B$ then $\sum_{n=1}^\infty \beta_n(x) b_n$ converges unconditionally to x if and only if for every sequence (a_n) of 0's and 1's there exists $y \in B$ with $\beta_n(y) = a_n \beta_n(x)$ for all n. This theorem improves previous results of Kadec and Pelczynski.

Similar results are obtained in the context of biorthogonal decompositions of a Banach space into separable subspaces.

1. Preliminaries. We follow the notation of [2] for the most part, and we also refer the reader to [2] for various results concerning unconditional convergence. We recall that a sequence of pairs $\{(b_n, \beta_n)\}$ is called a *biorthogonal sequence* in the Banach space B if for all m and n, $b_m \in B$, $\beta_n \in B^*$, and $\beta_m(b_n) = \delta_{mn}$; (β_n) is said to be *total* (in B) if given $x \in B$ with $\beta_n(x) = 0$ for all n, then $x = 0$. Finally, we denote the space of all bounded scalar-valued sequences by l_∞.

2. The Main Result. We first need the following lemma, due to Seever [8]:

LEMMA 1. *Let X be a Banach space and $T: X \to l_\infty$ be a bounded linear map such that for every $a \in l_\infty$ with $a_n = 0$ or 1 for all n, there exists $x \in X$ with $Tx = a$. Then $T(X) = l_\infty$.*

Proof. Our hypotheses imply that T has dense range; thus it is enough to show that T has closed range. If not, then T^* does not have closed range, so there exists a sequence (γ_n) in l_∞^* with $\|\gamma_n\| \to \infty$ and $\|T^*(\gamma_n)\| = 1$ for all n. But if $a \in l_\infty$ and $a_n = 0$ or 1 for all n, then choosing $x \in X$ with $Tx = a$, we have that

$$\sup_n |\gamma_n(a)| = \sup_n |T^*\gamma_n(x)| \leqq \|x\| < \infty .$$

Thus identifying l_∞ with $C(\beta N)$ (the space of continuous scalar-valued functions on the Stone-Čech compactification of N) and each γ_n with a complex regular Borel measure on βN, we have by a theorem of Dieudonne [3] (c.f. also the *Correction*, pp. 311–313 of [7]) that