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RELATIONS NOT DETERMINING THE STRUCTURE OF L

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A relation S is said to be determined up to isomorphism by relations R with respect to a theory K if for all models $\mathfrak{A}_1, \mathfrak{A}_2$ of $K, \mathfrak{A}_1$ restricted to R is isomorphic to $\mathfrak{A}_2$ restricted to R implies $\mathfrak{A}_1$ is isomorphic to $\mathfrak{A}_2$. In this paper simple necessary conditions for S to be determined up to isomorphism by R are given. These are applied in set theory to show there are (nonstandard) models of set theory with isomorphic ordinals and nonisomorphic constructible sets. The isomorphism on the ordinals may be taken to preserve many familiar arithmetic functions on the ordinals as addition, multiplication and exponentiation.

In this paper we show that the structure of the constructible sets of a model of set theory is not determined by the order-type of its ordinals, or, in fact, by its ordinals with various familiar arithmetic functions. This is shown by exhibiting (nonstandard) models of set theory with isomorphic ordinals and nonisomorphic constructible sets. The isomorphism on the ordinals may be taken to preserve familiar arithmetic functions.

These results are obtained by the use of certain simple general model-theoretic results developed in § 2. We define a relation S to be determined up to isomorphism by a set of relations R with respect to a theory K if $\langle A, R_A, S_A\rangle \models K$, $\langle B, R_B, S_B\rangle \models K$, $\langle A, R_A\rangle \approx \langle B, R_B\rangle$ implies $\langle A, R_A, S_A\rangle \approx \langle B, R_B, S_B\rangle$. We then give two simple sufficient conditions for S not to be determined up to isomorphism by R wrt K. Firstly, by a modification of a model-theoretic proof of Beth's theorem relating implicit and explicit definability, we show S is not determined up to isomorphism by R if there is a sentence σ such that the consequences about R of K, $K \cup \{\sigma\}$, $K \cup \{\neg \sigma\}$ are all the same. Using this, we show S is not determined up to isomorphism by R wrt K if there is a model $\mathfrak{A}$ of K in which the truth set of $\mathfrak{A}$ is not Turing-reducible to K join the truth set of $\mathfrak{A}$ restricted to R.

After illustrating simple applications of these results in § 2, we turn to the main set theory results in § 3. We observe that for any model $\mathfrak{A}$ of set theory, $\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}\rangle \equiv \langle \omega^{\omega}, <\rangle$ which has recursive truth set [9] and that the truth set of $\langle On_{\mathfrak{A}}, <_{\mathfrak{A}}, \tilde{\varepsilon}_{\mathfrak{A}}, \cong_{\mathfrak{A}}\rangle$ is not recursive (where F is the map defined up Gödel from $On \rightarrow L$, $\alpha \tilde{\varepsilon} \beta$ if $F(\alpha) \varepsilon F(\beta)$, $\alpha \cong \beta$ if $F(\alpha) = F(\beta)$ [8; 16]. Using this we may by