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SOME RESULTS ON COMPLETABILITY IN COMMUTATIVE RINGS

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In this paper, R always denotes a commutative ring with identity. The ideal of nilpotents and the Jacobson radical of the ring R are denoted by $N(R)$ and $J(R)$, respectively. The vector $[a_1, \cdots, a_n]$ is called a primitive row vector provided $1 \in (a_1, \cdots, a_n)$; a primitive row vector $[a_1, \cdots, a_n]$ is called completable provided there exists an $n \times n$ unimodular matrix over R with first row $a_1, \cdots, a_n$. A ring R is called a B-ring if given a primitive row vector $[a_1, \cdots, a_n], n \geqq 3$, and

$$(a_1, \cdots, a_{n-2}) \nsubseteq J(R),$$

there exists $b \in R$ such that $1 \in (a_1, \cdots, a_{n-2}, a_{n-1} + ba_n)$. Similarly, R is defined to be a Strongly B-ring (SB—ring), if $d \in (a_1, \cdots, a_n), n \geqq 3$, and $(a_1, \cdots, a_{n-2}) \nsubseteq J(R)$ implies that there exists $b \in R$ such that $d \in (a_1, \cdots, a_{n-2}, a_{n-1} + ba_n)$.

In this paper it is proved that every primitive vector over a B-ring is completable. It is shown that the following are B-rings: π-regular rings, quasi-semi-local rings, Noetherian rings in which every (proper) prime ideal is maximal, and adequate rings. In addition it is proved that $R[X]$ is a B-ring if and only if R is a completely primary ring. It is then shown that the following are SB-rings: quasi-local rings, any ring which is both an Hermite ring and a B-ring, and Dedekind domains. Finally, it is shown that $R[X]$ is an SB-ring if and only if R is a field.

2. B-rings.

LEMMA 2.1. *Let R be a ring with $A \subseteqq J(R)$, A an ideal of R. Then R is a B-ring if and only if R/A is a B-ring.*

Proof. Necessity: Let R be a B-ring and let

$$(1 + A) \in (a_1 + A, \cdots, a_n + A), n \geqq 3$$

and

$$(a_1 + A, \cdots, a_{n-2} + A) \nsubseteq J(R/A) = J(R)/A ,$$

where $a_i \in R, i = 1, \cdots n$. Then $1 + A = \sum_{i=1}^{n} a_i b_i + A, b_i \in R$; hence $[a_1, \cdots, a_n]$ is primitive. Since $(a_1, \cdots, a_{n-2}) \nsubseteq J(R)$, it follows that $[a_1 + A, \cdots, a_{n-2} + A, (a_{n-1} + ba_n) + A]$ is primitive for some $b \in R$. Therefore, R/A is a B-ring.

Sufficiency: Suppose R/A is a B-ring and suppose $[a_1, \cdots, a_n]$ is a