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GENERALIZED FINAL RANK FOR ARBITRARY LIMIT ORDINALS

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Let G be a p-primary Abelian group. The final rank of G can be obtained in two equivalent ways: either as $\inf_{n\in\omega}\{r(p^nG)\}$ where $r(p^nG)$ is the rank of p^nG; or as $\sup\{r(G/B)\mid B$ is a basic subgroup of $G\}$. In fact it is known that there exists a basic subgroup of G such that $r(G/B)$ is equal to the final rank of G. In this paper are displayed two appropriate generalizations of the above definitions of final rank, $r_\alpha(G)$ and $s_\alpha(G)$, where α is a limit ordinal. It is shown that the two cardinals $r_\alpha(G)$ and $s_\alpha(G)$ are indeed the same for any limit ordinal α. In this context one can think of the usual final rank as "ω-final rank".

The final rank of a p-primary Abelian group G is $\inf_{n<\omega}\{r(p^nG)\}$ where $r(p^nG)$ means the rank of p^nG. The same cardinal number is obtained by taking $\sup_{B\in\Gamma} r(G/B)$ where Γ is the set of all basic subgroups of G. In [1] we defined for limit ordinals α, $s_\alpha(G)=\inf_{\beta<\alpha} r(p^\beta G)$ and $r_\alpha(G)=\sup_{H\in\Gamma} r(G/H)$ where Γ is the set of all p^α-pure subgroups H of G such that G/H is divisible; it was shown that for accessible ordinals α that $r_\alpha(G)=s_\alpha(G)$. The proof given there strongly depended on the accessibility of α. In this paper it is proved that $r_\alpha(G)=s_\alpha(G)$ for any limit ordinal α, at the cost of a considerably more difficult argument.

Throughout we consider a reduced p-primary Abelian group G. We consider cardinal and ordinal numbers in the sense of von Neumann; that is, an ordinal number is a set, namely, the set of all smaller ordinals. Cardinal numbers are ordinal numbers that are not equivalent to any smaller ordinal. The cardinal number of a set Γ is denoted by $|\Gamma|$. The symbol ω denotes the first infinite ordinal. In general the notation and terminology is that of [2] or [3].

1. The lemmas.

Let τ be a limit ordinal. We define the final τ-rank of G in two ways, which we will then show are equivalent. Ordinary final rank as defined in [2] corresponds to final ω-rank.

DEFINITION.

(1) $s_\tau(G)=\inf_{\beta<\tau} r(p^\beta G[p])$.

(2) $r_\tau(G)=\sup\{r(G/H)\colon H\subseteqq G,\ G/H$ is divisible, and $0\to H\to G\to G/H\to 0$ represents an element of $p^\tau\operatorname{Ext}(G/H,H)\}$.

In [1] it is shown that $r_\tau(G)\leqq s_\tau(G)$. To show the converse we