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ON A PROBLEM OF DANZER

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By a Danzer set S we shall mean a subset of the n-dimensional Euclidean space R_n which has the property that every closed convex body of volume one in R_n contains a point of S. L. Danzer has asked if for $n \geqq 2$ there exist such sets S with a finite density. The answer to this question is still unknown. In this note our object is to prove two theorems about Danzer sets.

If Λ is a n-dimensional lattice, any translate $\Gamma = \Lambda + p$ of Λ will be called a grid Γ; Λ will be called the lattice of Γ and the determinant $d(\Lambda)$ of Λ will be called the determinant of Γ and will be denoted by $d(\Gamma)$. In § 2 we prove

THEOREM 1. For $n \geqq 2$, a Danzer set cannot be the union of a finite number of grids.

Let S be a Danzer set and $X > 0$ a positive real number. Let $N(S, X)$ be the number of points of S in the box $\max_{1 \leqq i \leqq n} |x_i| \leqq X$. Let $D(S, X) = N(S, X)/(2X)^n$. In § 3 we prove

THEOREM 2. There exist Danzer sets S with $D(S, X) = 0((\log X)^{n-1})$ as $X \to \infty$.

The case $n = 2$ of the theorem is known, although no proof seems to have been published. The referee has pointed out that a lower bound of 2 can easily be established for the density of a Danzer set in $n = 2$, but the authors are unaware of any further results in this direction.

2. Proof of Theorem 1. We shall assume throughout that $n \geqq 2$. It is obvious that if S is a Danzer set and T is a volume preserving affine transformation of R_n onto itself, then $T(S)$ is also a Danzer set.

Let $S_1, S_2, \cdots$ be a sequence of sets in R_n. Let S be the set of points X such that there exists a subsequence $S_{i_1}, S_{i_2}, \cdots$ of $\{S_r\}$ and points $X_{i_r} \in S_{i_r}$, such that $X_{i_r} \to X$ as $r \to \infty$. We write

$$S = \lim_{r \to \infty} S_r = \lim S_r .$$

LEMMA 1. *Let $\{S_r\}$ be a sequence of Danzer sets in R_n. Then $S = \lim S_r$ is also a Danzer set.*

Proof. Let K be a closed convex body of Volume 1. Then for