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AN ESTIMATE FOR WIENER INTEGRALS CONNECTED WITH SQUARED ERROR IN A FOURIER SERIES APPROXIMATION

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If a function $x(\sigma)$, $0 \leqq \sigma \leqq t$, is in Lip-α, $0 < \alpha < 1$, $x(0) = 0$ and if c_k $(k = 0, 1, 2, \cdots)$ are its Fourier coefficients with respect to the functions $\sqrt{2/t} \sin [\pi(k + \frac{1}{2})\sigma/t]$, then it is known [1, pp. 171–172] that

$$\sum_{k \geqq n} c_k^2 \leqq \frac{A}{(n + \frac{1}{2})^{2\alpha}} , \qquad n \geqq 0 \tag{1}$$

where A is a positive number not depending on n. We will show a connection between this estimate and an estimate for Wiener integrals. Let $E_w\{\ \}$ denote expectation on a Wiener process, that is, a Gaussian process with mean function zero, covariance function min (σ, τ), $0 \leqq \sigma, \tau \leqq t$ and sample functions $z(\sigma)$ with $z(0) = 0$.

THEOREM: Let $x(\sigma)$ be in $C[0, t]$ and let c_k be the Fourier coefficients of $x(\sigma)$ with respect to the normalized eigenfunctions associated with $\min(\sigma, \tau)$. That is

$$c_k = \sqrt{\frac{2}{t}} \int_0^t x(\sigma) \sin [\pi(k + \tfrac{1}{2})\sigma/t] d\sigma .$$

Let $0 < \alpha < 1$. Then estimate (1) is a necessary and sufficient condition for the estimate

$$e^{-(B/2)\nu^{1-\alpha}} \leqq \frac{E_W\left\{e^{-(\nu/2)\int_0^t [z(\sigma) - x(\sigma)]^2 d\sigma}\right\}}{E_W\left\{e^{-(\nu/2)\int_0^t z^2(\sigma) d\sigma}\right\}} \tag{2}$$

for all positive ν, where B is a positive number not depending on ν.

Proof. From Cameron and Donsker's proof of a lemma [2, p. 27–28], we have that, for the case $\rho_k = [\pi(k + \frac{1}{2})/t]^2$, the right side of (2) equals

$$e^{-\nu/2 \sum_{k=0}^{\infty} \frac{c_k^2 \rho_k}{\rho_k + \nu}} .$$

Hence estimate (2) holds if and only if

$$\sum_{k=0}^{\infty} \frac{c_k^2 \rho_k}{\rho_k + \nu} \leqq \frac{B}{\nu^\alpha} \tag{3}$$