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ON THE OTHER SET OF THE BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE LAGUERRE POLYNOMIALS

TILAK RAJ PRABHAKAR

Recently Konhauser considered the biorthogonal pair of polynomial sets $\{Z_n^\alpha(x;k)\}$ and $\{Y_n^\alpha(x;k)\}$ over $(0,\infty)$ with respect to the weight function $x^\alpha e^{-x}$ and the basic polynomials x^k and x. For the polynomials $Y_n^\alpha(x;k)$, a generating function, some integral representations, two finite sum formulae, an infinite series and a generalized Rodrigues formula are obtained in this paper.

Biorthogonality and some other properties of $Z_n^\alpha(x;k)$ and $Y_n^\alpha(x;k)$ for any positive integer k were discussed by Konhauser ([1], [2]). For $k=2$, the polynomials were discussed earlier by Preiser [4]. For $k=1$, the polynomials $Y_n^\alpha(x;k)$, as also $Z_n^\alpha(x;k)$, reduce to the generalized Laguerre polynomials $L_n^\alpha(x)$.

In a recent paper [3], we obtained generating functions and other results for the polynomials $Z_n^\alpha(x;k)$ in x^k. The present paper is concerned only with the polynomials $Y_n^\alpha(x;k)$ in x which form the other set of the biorthogonal pair. The results of the paper reduce, when $k=1$, to some standard properties of $L_n^\alpha(x)$. Simplicity of the procedure for deriving the generating relation (2.1) which may be regarded as our principal result, seems to be of some passing interest.

2. A generating function for $Y_n^\alpha(x;k)$.

We begin with the contour integral representation [2, (26)]

$$Y_n^\alpha(x;k) = (k/2\pi i)\int_C e^{-xt}(t+1)^{\alpha+kn}[(t+1)^k-1]^{-(n+1)}\,dt \tag{2.1}$$

where we take C as a closed contour enclosing $t=0$ and lying within $|t|<1$. If we make the substitution $u = 1-(t+1)^{-k}$, we get another integral representation for $Y_n^\alpha(x;k)$, viz.

$$Y_n^\alpha(x;k) = (2\pi i)^{-1}\int_{C'}(1-u)^{-(\alpha+1)/k}\exp[x\{1-(1-u)^{-1/k}\}]u^{-n-1}\,du \tag{2.2}$$

C' being a circle with centre $u=0$ and a small radius. By standard arguments of complex analysis we obtain the generating relation

$$\sum_{n=0}^{\infty} Y_n^\alpha(x;k)u^n = (1-u)^{-(\alpha+1)/k}\exp[x\{1-(1-u)^{-1/k}\}] \tag{2.3}$$

for $\mathrm{Re}(\alpha+1)>0$, $|u|<1$ and positive integers k.