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VARIETIES OF IMPLICATIVE SEMILATTICES

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The main purpose of this paper is to investigate properties of the lattice of subvarieties of the variety of implicative semilattices. Also the distinct compositions of the operators of taking homomorphic images, subalgebras, and products of classes of implicative semilattices are determined.

A class K of similar universal algebras [2, pp. 33–34] is called a variety provided K consists of all the algebras which satisfy some set of identities. If K is a variety and $K' \subseteqq K$, then K' is called a subvariety of K provided K' is itself a variety. The subvarieties of a given variety form a lattice when ordered by inclusion. A basic theorem of Birkhoff [1] states that a class of algebras K is a variety if and only if K is closed under the taking of homomorphic images, subalgebras, and direct products.

An implicative semilattice is an algebra $\langle L; \wedge, *\rangle$ where $\langle L; \wedge\rangle$ is a semilattice, and $*$ is a binary operation such that $x \wedge y \leqq z$ if and only if $x \leqq y * z$ (here $w \leqq u$ means $w \wedge u = w$). Every implicative semilattice has a largest element which we denote by 1. Monteiro [4] has given a set of equational axioms for implicative semilattices thus showing that the class of implicative semilattices is a variety. In this paper we consider the lattice of subvarieties of this variety. We shall denote the variety of all implicative semilattices by I.

An ideal K of an implicative semilattice L is a subset of L such that $x \in K$ whenever $x \leqq y \in K$. A filter J of L is a subset of L such that $x \wedge y, z \in J$ whenever $x, y \in J, z \in L$ and $x \leqq z$. It is shown in [5] that filters are related to homomorphisms for implicative semilattices in the same way they are for boolean algebras. In particular, if by the kernel of a homomorphism we mean the pre-image of the greatest element of the range, then the kernel of any homomorphism is a filter, every filter is the kernel of a homomorphism, and the congruence relation and quotient algebra determined by the homomorphism are also determined by the kernel of the homomorphism. Also, in [6] it is shown that the lattice of filters and therefore the lattice of congruence relations of an implicative semi-lattice is distributive. Thus we are able to make use of the results and techniques of Jónsson [3].

In § 2 we determine several elementary properties of subdirectly irreducible implicative semilattices. Here it is also shown that the variety of implicative semilattices is generated by its finite members.