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BV-FUNCTIONS ON SEMILATTICES

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It has been shown that the cone C of completely monotonic functions on a commutative semigroup G with identity induces a vector lattice ordering on the vector space $E = C - C$ spanned by C. An intrinsic characterization of the absolute value of the functions in E is desirable. In the present work we offer such a characterization when each member of G is idempotent, i.e. G is a semilattice. A notion of variation and bounded variation (BV) of arbitrary functions on G is introduced. We show that E is precisely the family of BV-functions and that if $f \in E$, then our concept of variation of f agrees with the usual absolute value as given by $f \vee (-f)$.

In case the natural order on G is linear, then C is the cone of nonnegative, nondecreasing functions and our notions of variation and bounded variation reduce to the classical concepts. More generally, we show that our variation reduces (not trivially) to the variation defined by Birkhoff [2] for BV-valuations on a distributive lattice with largest element.

1. Completely monotonic functions on semilattices.

In order to set the stage for our investigations, it will be necessary for us to recall [cf. 1] how the integral representation of a completely monotonic function simplifies when the underlying semigroup is a semilattice.

If f is a real-valued function defined on a commutative semigroup G with identity 1, then the difference operators Δ_n, for n a nonnegative integer, are defined inductively by $\Delta_0 f(a) = f(a)$, and $\Delta_n f(a; b_1, \cdots, b_n) = \Delta_{n-1} f(a; b_1, \cdots, b_{n-1}) - \Delta_{n-1} f(ab_n; b_1, \cdots, b_{n-1})$. The function f is said to be *completely monotonic* if $\Delta_n f(a; b_1, \cdots, b_n) \geqq 0$ for all choices of $a, b_1, \cdots, b_n \in G$. Let $C = C(G)$ denote the family of all completely monotonic functions on G and $C_1 = \{f \in C: f(1) = 1\}$. Then C is a convex cone with base [9] C_1, in the linear space R^G of all real valued functions on G. If we equip R^G with the topology of pointwise convergence, then the span, $E = C - C$, of C becomes a locally convex linear topological space and C_1 is compact. From [5] we see that C_1 is an r-simplex, i.e. every $f \in C_1$ admits a unique representing measure which is supported by the extremal points ($\operatorname{ext} C_1$) of C_1, and $\operatorname{ext} C_1$ is closed. A nontrivial homomorphism from G into the multiplicative semigroup of real numbers in the closed unit interval is called an *exponential*. The set of exponentials on G