PACIFIC JOURNAL OF MATHEMATICS
Vol. 38, No. 1, 1971

ON THE RANGE OF A DERIVATION

J. P. WILLIAMS

A derivation on an algebra $\mathscr{A}$ is a linear transformation δ on $\mathscr{A}$ with the property $\delta(XY) = X\delta(Y) + \delta(X)Y$ for all $X, Y \in \mathscr{A}$. If $\mathscr{A} = \mathscr{B}(\mathscr{H})$ is the Banach algebra of all bounded linear operators on a complex separable infinite-dimensional Hilbert space $\mathscr{H}$ then it is known that every derivation δ on $\mathscr{A}$ is inner, that is, there is a bounded operator A on $\mathscr{H}$ such that $\delta(X) = AX - XA = \delta_A(X)$ for all $X \in \mathscr{B}(\mathscr{H})$. (See [8].) In the present note simple necessary and sufficient conditions are obtained that (i) the range $\mathscr{R}(\delta_A)$ be dense in the weak and ultraweak operator topologies; (ii) the norm closure of the range contain the ideal $\mathscr{K}$ of compact operators on $\mathscr{H}$, (iii) the set of commutators $BX - XB$ where B belongs to the C^*-algebra generated by A and X is arbitrary be weakly or ultraweakly dense in $\mathscr{B}(\mathscr{H})$. The commutant of the range of a derivation is also computed and it is shown that the ranges of any two nonzero derivations have nonzero intersection.

1. If A and B are bounded operators on $\mathscr{H}$ then the identities $\delta_A + \delta_B = \delta_{A+B}$, $\delta_A\delta_B - \delta_B\delta_A = \delta_{AB-BA}$ show that the sum and Lie product of two (inner) derivations is a derivation. However the product $\delta_A\delta_B$ is a derivation only in the trivial cases:

THEOREM 1. *Let $A, B \in \mathscr{B}(\mathscr{H})$. The $\delta_A\delta_B$ is a derivation if and only if A or B is a scalar multiple of the identity operator.*

Proof. Let $\delta = \delta_A\delta_B$. Then

$$\begin{aligned}\delta(XY) &= \delta_A(\delta_B(XY)) = \delta_A(X\delta_B(Y) + \delta_B(X)Y) \\ &= X\delta_A\delta_B(Y)) + \delta_A(X)\delta_B(Y) + \delta_A(\delta_B(X))Y + \delta_B(X)\delta_A(Y) \\ &= X\delta(Y) + \delta(X)Y + \delta_A(X)\delta_B(Y) + \delta_B(X)\delta_A(Y)\ .\end{aligned}$$

Therefore δ is a derivation if and only if

$$\delta_A(X)\delta_B(Y) + \delta_B(X)\delta_A(Y) = 0 \tag{1}$$

for all $X, Y \in \mathscr{B}(\mathscr{H})$. Replacing X by XZ in (1) we get

$$0 = X\delta_A(Z)\delta_B(Y) + \delta_A(X)Z\delta_B(Y) + X\delta_B(Z)\delta_A(Y) + \delta_B(X)Z\delta_A(Y)$$

so that

$$\delta_A(X)Z\delta_B(Y) + \delta_B(X)Z\delta_A(Y) = 0 \tag{2}$$