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FATOU'S LEMMA IN NORMED LINEAR SPACES

STEPHEN SCHEINBERG

This note presents a generalization of Fatou's lemma to arbitrary normed linear spaces. Several examples illustrate the situations in which this notion is meaningful. The main theorem gives an abstract characterization of the Fatou property. In particular this resolves the case of any reflexive space. An example shows that Fatou's lemma may fail even for uniform convergence in a normed algebra of continuous functions.

Frequently in analysis one obtains a function by a limiting process which is weaker (less demanding) than convergence in the norm. For example, a continuous function may be obtained as the point-wise, but not necessary uniform, limit of other continuous functions. Even though the limit is not a norm limit, one may still need to know that the norm of the limit function is no greater than the norms of the approximating functions. The classical case is, of course, Fatou's lemma: if $f_n \to f$ pointwise, then

$$\int |f| \leqq \liminf \int |f_n| .$$

Another common situation is this. A subspace $A \subseteq C(X)$ is given which has a norm, $||f|| \geqq \sup |f|$. If $f_n \to f$ pointwise (or uniformly), does it follow that $||f|| \leqq \liminf ||f_n||$? The answer is "yes" quite often, but can be "no," even when A is a subalgebra.

Motivated by a wide variety of examples, I wish to consider the following general situation. Throughout this paper A will be a normed linear space, not necessarily complete, and $\mathscr{T}$ will be a locally convex Hausdorff topology on A which is weaker (coarser) than the norm topology. Say that *Fatou's lemma holds for A relative to $\mathscr{T}$* if whenever $a_\beta \underset{\beta}{\to} a$ in $\mathscr{T}$, it follows that $||a|| \leqq \lim_\gamma \inf_{\beta \geqq \gamma} ||a_\beta||$. It is usually easier to apply the equivalent condition stated in the following proposition.

PROPOSITION. *Fatou's lemma holds for A relative to $\mathscr{T} \Leftrightarrow$ whenever $a_\beta \underset{\beta}{\to} a$ in $\mathscr{T}$, it follows that $||a|| \leqq \sup_\beta ||a_\beta||$.*

Proof. $\Rightarrow$ is obvious, since $\sup \geqq \liminf$. For $\Leftarrow$, let $a_\beta \underset{\beta}{\to} a$ in $\mathscr{T}$ be given. We may assume $\lim_\gamma \inf_{\beta \geqq \gamma} ||a_\beta|| = L < \infty$, for otherwise there is nothing to prove. Fix $\varepsilon > 0$. Given γ, $\exists \beta \geqq \gamma$ with $||a_\beta|| \leqq L + \varepsilon$. Write this β as $\beta(\gamma)$ and we obtain $a_{\beta(\gamma)} \underset{\gamma}{\to} a$ with $\sup_\gamma ||a_{\beta(\gamma)}|| \leqq L + \varepsilon$. By hypothesis, then $||a|| \leqq L + \varepsilon$. Now let $\varepsilon \to 0$.