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SHARP ESTIMATES OF CONVOLUTION TRANSFORMS IN TERMS OF DECREASING FUNCTIONS

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Let $f*g$ denote the convolution transform of two Lebesgue measurable functions on the real line defined by the equation

$$(f*g)(x)=\int_{-\infty}^{+\infty}f(t)g(x-t)dt\ .$$

We get best possible upper and lower estimates for the expression

$$\sup_{\substack{f_i\sim g_i^*\\ |E|\leqq u}}\int_E |(f_1*\cdots *f_n)(x)|^p d(x)$$

where $p=1$ and 2, with applications to Fourier transform inequalities. Here g_i^* are preassigned decreasing functions and the symbol $f_i\sim g_i^*$ means

$$|\{x: |f_i(x)|>y\}|=|\{x: g_i^*(x)>y\}| \text{ for all } y.$$

0. Introduction. In order to formulate the general problem, we remember the Hardy and Littlewood estimate [4, page 130, Theorem 6.8] of the L_q-norm ($q\geqq 2$) for the Fourier coefficients $c_n=\int_{-\pi}^{\pi}f(t)e^{int}$ of f in terms of the decreasing rearrangement g^* of $|f|$:

$$||c_n||_q\leqq A_q\int_0^{2\pi}(g^*(x))^q x^{q-2}d(x) \tag{0.1}$$

where the constant A_q depends only on q. As one might expect, the same theorem [Theorem E, this paper] holds for the Fourier transform $\mathfrak{F}(f)$, i.e.

$$\int_{-\infty}^{+\infty}|\mathfrak{F}(f)|^q\leqq c(q)\int_0^{\infty}(g^*(x))^q x^{q-2}d(x) \tag{0.2}$$

holds for $q\geqq 2$, where g^* is the decreasing rearrangement of $|f|$. It is very remarkable that for (0.1), this is the best possible estimate in terms of g^*, i.e. independent of signs and arrangement of f, since for prescribed g^* the left side always reaches the right side for some suitable f and this same result is valid for (0.2). Thus, Hardy and Littlewood were able to determine the

$$\sup ||\mathfrak{F}(f)||_q\ ,$$

where f varies over all functions with the same g^*. If $q=2k$ (k