HOMOLOGICAL ALGEBRA OF STABLE HOMOTOPY RING π_* OF SPHERES

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The stable homotopy groups are studied as a graded ring π_* via homological algebra. The main object is to show that the projective (and weak) dimension of a finite type π_*-module is ∞ unless the module is free. As a corollary, a partial answer to Whithead's corollary to Freyd's generating hypothesis is obtained.

1. **Introduction and statement of main results.** It is well-known that the stable homotopy groups of spheres form a commutative graded ring π_* [20]. This paper is our first effort towards the investigation of the homological properties of the stable homotopy ring π_*. In this paper we have completed the computations of all the homological numerical invariants of finitely generated type. The nonfinitely generated type will be taken up in forth-coming papers.

The paper is organized as follows: The introduction is § 1. In § 2 we give a brief exposition about the theory of graded rings, which are needed in later sections. § 3 is primarily a preparation for § 4. Here the finitistic global dimensions of the "p-primary component" Λ_p of π_* (precisely, Λ_p is a ring obtained by localizing π_* at a maximal ideal) are computed, and a geometric realization of Λ_p is constructed. § 4 is the mainbody of this paper, here we prove Theorem 2 and derive from it Theorems 1, 3, and 4. We would like to suggest that the reader, after § 1, go directly to § 4 and refer to the rest of the sections when necessary.

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The main results are:

THEOREM 1. *Let A be a locally finitely generated* (i.e., there are only finitely many generators at each degree) *π_*-module having finite projective, as well as weak, dimension. Then A is a free π_*-module and hence is realizable as a stable homotopy module $\pi_*(Y)$ by a wedge Y of spheres.*

In [7] Freyd propose a conjecture, known as the generating hypothesis, which asserts that a map between finite CW-complexes, which induces the zero map on stable homotopy groups, is stably null-homotopic. A consequence of this conecture, due to G. Whitehead, asserts that the finitely generated stable homotopy module $\pi_*(X)$ of