

COUNTEREXAMPLES TO A CONJECTURE OF G. N. DE OLIVEIRA

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G. N. de Oliveira gives the following conjecture.

CONJECTURE. Let A be an $n \times n$ doubly stochastic irreducible matrix. If n is even, then $f(z) = \text{perm}(Iz - A)$ has no real roots; if n is odd, then $f(z) = \text{perm}(Iz - A)$ has one and only one real root.

In this paper we give counter examples to this conjecture.

Results:

EXAMPLE 1. Let

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

$f(z) = \text{perm}(Iz - A)$ is such that $f(0) < 0$ and $f(1) > 0$. Consider $f(z) \cdot (z - 1) = g(z)$. Note that $g(0) > 0$ and since there is a ξ ($0 < \xi < 1$) for which $f(\xi) > 0$ we see that $g(\xi) < 0$. Now consider

$$B(\varepsilon) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & -\varepsilon & \varepsilon \\ 0 & 0 & \varepsilon & 1 - \varepsilon \end{bmatrix}.$$

If $0 \leq \varepsilon \leq \frac{3}{4}$, $B(\varepsilon)$ is doubly stochastic. Further if $g_\varepsilon(z) = \text{perm}[Iz - B(\varepsilon)]$ then for each z , $g_\varepsilon(z) = \lim_{\varepsilon \rightarrow 0} g_\varepsilon(z)$. Since $g_\varepsilon(0) > 0$ for each ε and $g(\xi) = \lim_{\varepsilon \rightarrow 0} g_\varepsilon(\xi) < 0$ we see that for sufficiently small ε , say ε_0 , $g_{\varepsilon_0}(z)$ has a real root and $B(\varepsilon_0)$ is irreducible. This yields the counter-example. Note also that $g_{\varepsilon_0}(z) > 0$ for $z > 1$ [see 1], hence $g_{\varepsilon_0}(z)$ has at least two real roots.

EXAMPLE 2. For simplification let $B(\varepsilon_0) = B$ and $g_{\varepsilon_0}(z) = g(z)$. Recall

- (a) $g(0) > 0$ and
- (b) $g(\xi) < 0$. By direct calculation we see that
- (c) $g(1) > 0$ and hence for some η , $\xi < \eta < 1$
- (d) $g(\eta) > 0$.

Now consider $f(z) = g(z) \cdot (z - 1)$. Note that

- (a) $f(0) < 0$
- (b) $f(\xi) > 0$