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INEQUALITIES FOR POSITIVE INTEGRAL OPERATORS

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The aim of this paper is to study integral inequalities of the following form, where T is an integral operator with non-negative kernel:

$$\int_a^b |Tf|^p |f|^q d\mu \leqq K\left\{\int_a^b |f|^r d\mu\right\}^{(p+q)/r} .$$

Classical examples of such inequalities include Hardy's inequality and Opial's inequality. Our main result (Theorem 1) is a minimax characterization of the best constants in such inequalities, under the condition that $1 \leqq p + q \leqq r$. This theorem allows us to deduce certain facts concerning the uniqueness of the extremal functions. We then apply these results to the explicit computation or estimation of the best constants in inequalities of the form:

$$\int_a^b |y(x)|^p |y''(x)|^q dx \leqq K\left\{\int_a^b |y''(x)|^r dx\right\}^{(p+q)/r} .$$

where $y(a) = y(b) = 0$.

In a previous paper [7], we showed that, if T is a compact mapping from L^r into L^s $(s = pr/(r - q))$, and $0 < p$, $0 \leqq q < r$ and $1 < r$, then the best constant in such an inequality is the largest eigenvalue of a certain nonlinear integral equation, the "variational equation".

The method which we develop here allows us to show that, in fact, the variational equation has at most one eigenvalue, if $1 \leqq p + q \leqq r$, (and exactly one if $1 \leqq p + q < r$, and $0 \leqq (p - 1)r + q$). This means that it is not, a priori, necessary to know of the existence of a solution to the variational equation as was the case in [7]. Our theorem is thus more closely related to the techniques used by Beesack in [2], [3], and to the results of Wilf ([12], p. 70) and Tomaselli [11]. The uniqueness results we obtain give us the opportunity to clear up some confusion concerning the results in the paper of Boyd and Wong [8].

One advantage of our approach via integral operators is that we obtain a uniform treatment of inequalities involving functions and derivatives higher than the first, which would not be the case if we reduced immediately to differential operators. The existence and uniqueness theorems which result may be of some interest in the theory of non-linear boundary value problems.

In § 6, we discuss a number of special situations in which the variational equation can be reduced to a simpler form, or the uni-