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BANACH ALGEBRAS WHICH ARE IDEALS IN A BANACH ALGEBRA

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In this paper Banach algebras A which are ideals in a Banach algebra B are studied. The main results concern the relationship between the norms of A and B and the relationship between the closed ideals of A and B.

There are many examples of Banach algebras in analysis which are ideals in another Banach algebra. When G is a locally compact group, then the Segal algebras which are studied in H. Reiter's book [7] are ideals in $L^1(G)$. J. Cigler considers more general Banach algebras which are ideals in $L^1(G)$ in [2]. In the theory of operators on a Hilbert space $\mathscr{H}$, the C_p algebras discussed in [4, pp. 1088–1119] are ideals in the algebra of compact operators on $\mathscr{H}$ (C_1 is the ideal of trace class operators and C_2 the ideal of Hilbert-Schmidt operators). Also as we point out in § 4, every full Hilbert algebra is a dense *-ideal in a B^*-algebra.

When A is a Banach algebra which is an ideal in a Banach algebra B, we consider the relationship between the algebras A and B. First we prove that the norms of A and B are related by certain inequalities. As a consequence, if B is semi-simple, then A is a left and right Banach module of B [Theorem 2.3]. Also in this case our results show that A is an abstract Segal algebra with respect to B as defined by J. T. Burnham in [1]. Secondly we relate the closed left and right ideals of A to those of B. Of special interest here is the case where A contains a bounded approximate identity of B [Theorem 3.4]. Finally in § 4 we consider the special case where A is a *-ideal in a B^*-algebra B. The results of this section apply to full Hilbert algebras.

1. Preliminaries and notation.

When B is any Banach algebra, we denote the Banach algebra norm on B by $\|\cdot\|_B$. If M is a closed left ideal in the Banach algebra B, $B - M = \{b + M \mid b \in B\}$ is the quotient module B modulo M. $B - M$ is normed by the norm

$$\|b + M\|'_B = \inf\{\|b - m\|_B \mid m \in M\} .$$

Throughout this paper A is a given Banach algebra. We always use the term "ideal" to mean two-sided ideal. A is usually an ideal in a Banach algebra B. In this case when E is a subset of A, $\mathrm{cl}(E)$ is the closure of E in B.