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TOTALLY REAL REPRESENTATIONS AND REAL FUNCTION SPACES

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Let G be a locally compact group. The notion of "totally real" unitary representation of G is defined and investigated in §1. In particular, if K is a compact subgroup of G, it is shown that every closed G-invariant subspace of $L_2(G/K)$ is spanned by real-valued functions if, and only if, $KgK=Kg^{-1}K$ for every $g \in G$. In §2 the coset space $X=G/K$ is specialized to a Riemannian symmetric space, where the double coset condition is replaced by a simple Weyl group condition.

0. **Introduction.** Let X be a riemannian symmetric space of compact type and G its largest connected group of isometries. In his 1929 paper [1] on class 1 representations, É. Cartan showed that the symmetry of X sends every uniformly closed G-invariant function space on X to its complex conjugate. Starting from the point of view of algebras, Mirkil and de Leeuw [4] showed that every rotation invariant function algebra on the sphere $S^n(n \geqq 2)$ was spanned by real-valued functions, hence (Stone-Weierstrass theorem) that such an algebra necessarily was all continuous functions on S^n, all continuous functions on real projective n-space, or just the constants-that state of affairs is quite different from the case $n = 1$. When the rotation group $SO(n+1)$ contains the symmetry of S^n, i.e. when n is even, Cartan's result mentioned above implies reality of such function algebras. The published Mirkil-de Leeuw argument rests rather on the fact that the spherical harmonics are real-valued.

The Cartan and Mirkil-de Leeuw results were unified when I. Glicksberg and one of us found a general result [12, Theorem 2.1] on G-invariant function spaces on compact symmetric spaces, formulated in terms of the double coset condition mentioned in the Abstract. One of us then translated the double coset condition into an easily-checked Weyl group condition [12, Theorem 5.1] and extended the Mirkil-de Leeuw result on function algebras [12, Theorem 7.1]. That translation made essential use of É. Cartan's classification of symmetric spaces, and was later freed of the classification by J. A. Tirao [9].

We discuss this circle of problems for coset spaces $X = G/K$, G locally compact and K compact. Although the idea is very much the same, the proofs are more streamlined and are freed of many differentiability and compactness restrictions. Until now, however, our only significant applications are to Riemannian symmetric spaces.