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ON BLASCHKE PRODUCTS OF RESTRICTED GROWTH

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Let $\mathscr{I}$ denote the class of Blaschke products $B(z, \{z_n\})$ such that

$$\int_0^{2\pi} (\log |B(re^{i\theta}, \{z_n\})|)^2 d\theta$$

is bounded for $0 < r < 1$. The distributions of the zeros of Blaschke products in $\mathscr{I}$ are examined, and extensions are made to earlier results of MacLane and Rubel.

1. Let $\{z_n\}$ be a nonempty sequence of nonzero complex numbers in $D(0, 1) = \{z: |z| < 1\}$. Then $\{z_n\}$ is a Blaschke sequence if and only if

$$\sum_n (1 - |z_n|) < \infty . \tag{1.1}$$

If (1.1) holds then the Blaschke product

$$B(z, \{z_n\}) = \prod_n \frac{\bar{z}_n}{|z_n|}\left(\frac{z_n - z}{1 - z\bar{z}_n}\right) \tag{1.2}$$

represents a function $B(z, \{z_n\})$ regular in $D(0, 1)$. It is well-known that $|B(z, \{z_n\})| < 1$ when $z \in D(0, 1)$, and that

$$\lim_{r \to 1-0} B(re^{i\theta}, \{z_n\})$$

exists and has modulus 1 for almost every θ in $[0, 2\pi]$.

Let

$$I(r, \{z_n\}) = \frac{1}{2\pi}\int_0^{2\pi} (\log |B(re^{i\theta}, \{z_n\})|)^2 d\theta . \tag{1.3}$$

MacLane and Rubel [3] have considered the class $\mathscr{I}$ of Blaschke products for which $I(r, \{z_n\})$ is bounded on $[0, 1)$. They have shown that a necessary and sufficient condition for $B(z, \{z_n\})$ to belong to $\mathscr{I}$ is that $J(r, \{z_n\})$ is bounded on $[0, 1)$, where

$$J(r, \{z_n\}) = \sum_{k=1}^{\infty} k^{-2}\left|(r^k - r^{-k})\sum_{|z_n| \leq r} \bar{z}_n^k + r^k \sum_{|z_n| > r} (\bar{z}_n^k - z_n^{-k})\right|^2 . \tag{1.4}$$

In fact, the work of Rubel [4] shows that

$$J(r, \{z_n\}) = 2I(r, \{z_n\}) - 2\left\{N(r, \{z_n\}) \log r + \log \prod_{|z_n| > r} |z_n|\right\}^2 , \tag{1.5}$$

where $N(r, \{z_n\})$ denotes the number of elements in the sequence $\{z_n\}$