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SCHLICHT MAPPINGS AND INFINITELY DIVISIBLE KERNELS

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The purpose of this note is to give a simple condition which is sufficient for a function on a real interval to be the boundary value of a schlicht (univalent) analytic mapping of the upper half plane into itself. This condition leads to a simple transformation which takes (possibly) non-schlicht mappings into schlicht ones. The methods used have applications to probability theory as well; they yield an interesting class of infinitely divisible characteristic functions.

We shall require some facts about infinitely divisible kernels; for a detailed exposition see [5]. If I is a real interval we denote by $L_0(I)$ the set of all continuous complex valued functions which have compact support in I and whose integral over I vanishes. A continuous kernel $K(x, y)$ on $I \times I$ is said to be *conditionally positive definite on* I if

$$\iint_{I\times I} K(x, y)\phi(x)\bar{\phi}(y)dxdy \geqq 0 \tag{1}$$

for all functions $\phi \in L_0(I)$; it is said to be *positive definite on* I if (1) is satisfied for all continuous functions ϕ with compact support in I; it is said to be *infinitely divisible on* I if (for some fixed continuous determination of the argument) the kernel $K^{\alpha}(x, y)$ is positive definite for all $\alpha > 0$.

The connection among these concepts is that a continuous Hermitian kernel $K(x, y)$ with no zeroes is infinitely divisible on I if and only if (for some continuous determination of the argument) the kernel $\log K(x, y)$ is conditionally positive definite on I. If $K(x, y) > 0$ for all $x, y \in I$ there is, of course, no difficulty about determining the argument. Finally, the relevance of these notions to function theory is indicated by the following result [6], [4]. If f is a differentiable function we define $K_f(x, y) \equiv [f(x) - f(y)]/(x - y)$ and agree that $K_f(x, x) = f'(x)$.

THEOREM 1. *Let f be a continuously differentiable real valued function with positive derivative on a real interval I. The function f possesses an analytic continuation onto the upper half plane which maps the upper half plane into itself if and only if the kernel $K_f(x, y)$ is positive definite on I. This mapping is schlicht if and only if $K_f(x, y)$ is infinitely divisible on I.*