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INDICES FOR THE ORLICZ SPACES

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The determination of the function spaces X which are intermediate in the weak sense between L^p and L^q has been shown, by the author, to depend on a pair of numbers (α, β) called the indices of the space. The indices depend on the function norm of X and on the properties of the underlying measure space: whether it has finite or infinite measure, is non-atomic or atomic. In this paper, formulas are given for the indices of an Orlicz space in case the measure space is non-atomic with finite or infinite measure, or else is purely atomic with atoms of equal measure. The indices for an Orlicz space over a non-atomic finite measure space turn out to be the reciprocals of the exponents of the space as introduced by Matuszewska and Orlicz, and generalized by Shimogaki. Some new results concerning submultiplicative functions are used in the proof of the main result.

DEFINITIONS AND MAIN RESULT. We suppose that $(\Omega, \mathscr{T}, \mu)$ is a measure space which is measure-theoretically isomorphic to one of the following three possibilities: the positive reals $\boldsymbol{R}^+$ with Lebesgue measure, the interval $I = [0, 1]$ with Lebesgue measure or the positive integers $\boldsymbol{Z}^+$ with counting measure. We shall denote these standard spaces by S_i with $i = 0, 1, 2$ for $\boldsymbol{R}^+$, I and $\boldsymbol{Z}^+$ respectively. We shall write $\Omega^* = S_i$ if $(\Omega, \mathscr{T}, \mu)$ is isomorphic to S_i.

Suppose that ρ is a rearrangement-invariant function norm on the measurable functions $\mathscr{M}(\Omega, \mathscr{T}, \mu)$ on $(\Omega, \mathscr{T}, \mu)$, (see [2]). Then for each measurable f, $\rho(f) = \sigma(f^*)$ where f^* is the decreasing rearrangement of f onto $\boldsymbol{R}^+$, and σ is a rearrangement invariant norm on $\boldsymbol{R}^+$. We define $E_s f^*$ by $E_s f^*(t) = f^*(st)$ for $0 < s < \infty$, and then let

$$h(s) = \sup \{\sigma(E_s f^*): f \in \mathscr{M}(\Omega, \mathscr{T}, \mu), \sigma(f^*) \leqq 1\} .$$

It is known that the following limits exist (see Lemma 1, below)

$$\alpha = \lim_{s \to 0+} \frac{\log h(s)}{-\log s}, \ \beta = \lim_{s \to \infty} \frac{-\log h(s)}{\log s}$$

and these are called the indices of the space $L^\rho(\Omega, \mathscr{T}, \mu)$. Note that in fact α depends only on σ and on Ω^* so we will write $\alpha = \alpha(\sigma, \Omega^*)$ and $\beta = \beta(\sigma, \Omega^*)$. (A slightly different definition was used in case $\Omega^* = \boldsymbol{Z}^+$ when the indices were introduced in [2], but it is equivalent to the definition given here as Lemma 3 will show). It was shown in [2] that X is intermediate between L^p and L^q in the weak sense if and only if $1/p > \alpha$ and $\beta > 1/q$.