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ON ITERATED w^*-SEQUENTIAL CLOSURE OF CONES

R. D. MCWILLIAMS

In this paper it is proved that for each countable ordinal number $\alpha \geqq 2$ there exists a separable Banach space X containing a cone P such that, if J_X is the canonical map of X into its bidual X^{**}, then the αth iterated w^*-sequential closure $K_\alpha(J_X P)$ of $J_X P$ fails to be norm-closed in X^{**}. From such spaces there is constructed a separable space W containing a cone P such that if $2 \leqq \beta \leqq \alpha$, then $K_\beta(J_W P)$ fails to be norm-closed in W^{**}. Further, there is constructed a (non-separable) space Z containing a cone P such that if $2 \leqq \beta < \Omega$, then $K_\beta(J_Z P)$ fails to be norm-closed in Z^{**}.

1. If X is a real Banach space and Y a subset of X^{**}, let $K(Y)$ be the set of elements of X^{**} which are w^*-limits of sequences in Y. Let $K_0(Y) = Y$ and inductively let $K_\alpha(Y) = K(\cup_{\beta<\alpha} K_\beta(Y))$ for $0 < \alpha \leqq \Omega$, where Ω is the first uncountable ordinal. A *cone* in X is a subset of X which is closed under addition and under multiplication by nonnegative scalars. Our main theorem extends the result of [6] that if P is a cone in X, then $K_1(J_X P)$ must be norm-closed but $K_2(J_X P)$ can fail to be norm-closed in X^{**}. By contrast it is noted that if S is a compact Hausdroff space and $X = C(S)$ and $\alpha < \Omega$, then $K_\alpha(J_X X)$ is norm-closed, even though for example if S is compact, metric, and uncountable, then $K_\alpha(J_X X)$ is not w^*-sequentially closed. It is obvious that for each Banach space X and each subset Y of X^{**}, $K_\Omega(Y)$ is w^*-sequentially closed and hence norm-closed.

In [7] a Banach space X was exhibited such that $K_2(J_X X)$ is not norm-closed. Whether $K_\alpha(J_X X)$ can fail to be norm-closed for $2 < \alpha < \Omega$ is not known to the author. However, in the present paper it will be convenient to use constructions involving spaces studied in [7].

Section 2 is devoted to a useful relationship between w^*-sequential convergence and pointwise convergence of bounded sequences of functions, § 3 to further study of a space constructed in [7], and §§ 4 and 5 to preparation for and proof of the main theorems.

2. Let S be a compact Hausdorff space, $B(S)$ the Banach space of bounded real functions on S with the supremum norm, and $C(S)$ the closed subspace of $B(S)$ consisting of the continuous real functions on S. If A is a subset of $B(S)$, let $L(A)$ be the set of all pointwise limits of bounded sequences in A, and let $L_\alpha(A)$ be defined inductively by $L_0(A) = A$ and $L_\alpha(A) = L(\cup_{\beta<\alpha} L_\beta(A))$ for each ordinal α such that $0 < \alpha \leqq \Omega$.

If X is a norm-closed subspace of $C(S)$ and $z \in L_\Omega(X)$, then z is