PACIFIC JOURNAL OF MATHEMATICS
Vol. 38, No. 3, 1971

ON ALGEBRA ACTIONS ON A GROUP ALGEBRA

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A complete description of all algebra actions of the group algebra $L^1(K)$ on the group algebra $L^1(G)[M(G)]$ for locally compact Abelian groups K and G is presented. A fundamental algebra action of $L^1(K)$ on $L^1(G)$ is that induced by a continuous homomorphism $\theta: K \to G$ via a generalized convolution; such actions have been considered by Gelbaum in characterizing topological tensor products of group algebras. It is shown in this paper that conversely every algebra action of $L^1(K)$ on $L^1(G)[M(G)]$ is induced by a necessarily continuous homomorphism of K into the quotient of G by a compact subgroup. The analysis is based on a representation theorem for algebra actions on $L^1(G)$ for general locally compact group G. Namely, every algebra action of a Banach algebra C on $L^1(G)$ is the composition of a necessarily continuous cetral homomorphism Ψ of C into $M(G)$ and convolution in $M(G)$: $c \cdot a = \Psi(c) * a$ for all $c \in C$ and $a \in L^1(G)$. Applications to topological tensor products of group algebras are announced.

Let G and K be locally compact Abelian groups, and, let θ be a continuous homomorphism of K into G. Gelbaum [3] has observed that θ induces a module action of the group algebra $L^1(K)$ on the group algebra $L^1(G)$ via a "generalized convolution": if $c \in L^1(K)$ and $a \in L^1(G)$, then $c *_\theta a \in L^1(G)$ is given by

$$c *_\theta a(g) = \int_K c(k)a(g - \theta(k))dk, \qquad (g \in G).$$

Moreover, this module action is associative in the sense that

$$(c *_\theta a) * a' = c *_\theta (a * a') \quad (= a * (c *_\theta a')$$

by commutativity of $L^1(G)$), and, the action satisfies the inequality $\|c *_\theta a\|_1 \leqq \|c\|_1 \|a\|_1$. Hence $L^1(G)$ is an algebra over $L^1(K)$ and the action is continuous, i.e., $L^1(G)$ is an (isometric) Banach $L^1(K)$-algebra.

The question we pose is "Does the converse hold, i.e., if $L^1(G)$ is an algebra over $L^1(K)$ such that $\|c \cdot a\|_1 \leqq \|c\|_1 \|a\|_1$ for all $c \in L^1(K)$ and $a \in L^1(G)$, is the action of $L^1(K)$ on $L^1(G)$ induced by a continuous homomorphism of K into G?" We answer this question in full, the main result culminating in Corollary 3.2.

I would like to take this opportunity to thank Professor Bernard Gelbaum, who suggested the above problem, for his kind encouragement and stimulating discussions during the preparation of part of this work. The author also thanks the referee for his numerous