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THE CONVEX HULLS OF THE VERTICES OF A POLYGON OF ORDER n

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Dedicated to Richard Rado on his sixty-fifth birthday.

Let $\pi_n : A_1A_2 \cdots A_r$ be a polygon of real order n in real projective n-space L_n, $r \geqq n+3$, with the vertices $A_1, A_2, \cdots, A_r$. If H_{n-1} be a hyperplane for which $A_i \notin H_{n-1}$, $1 \leqq i \leqq r$, let $H(\pi_n)$ be the convex full of the set $\{A_1, A_2, \cdots, A_r\}$ defined in the affine space $L_n \backslash H_{n-1}$. This paper gives a classification of the combinatorial types of the sets $H(\pi_n)$ for a fixed π_n.

The restriction $r \geqq n+3$ is necessary because the classification is obtained from the properties of the polygon π_n associated with $H(\pi_n)$. Such a polygon is determined uniquely by its vertices if and only if $r \geqq n+3$ [2]. If $r = n+3$ and the points $A_1, A_2, \cdots, A_{n+3}$ are in general position there is exactly one polygon π_n with these vertices [1]. Thus the set of all $H(\pi_n)$, $r = n+3$, for which no vertex of π_n an interior point of $H(\pi_n)$ is also the set of convex polytopes with $n+3$ vertices in general position. Consequently the invariants which characterize the sets $H(\pi_n)$ can be used to characterize the convex polytopes with $n+3$ vertices in general position. The method used here is different from that used by M. A. Perles in his solution [3] of this problem.

The work is divided into three sections. The first contains definitions and known or easily proved results dealing with the polygons π_n. The second section develops theorems involving the convex hulls of the polygon vertices which are used to obtain the characterizing invariants in the final section. Except for the case in which H_{n-1} intersects π_n in n points the sets are characterized by a cycle of the type used in combinatorial analysis [4].

1. Preliminaries.

1.1. The subspace of the real projective n-space L_n spanned by the points or point sets $A, B, \cdots$ is denoted by $[A, B, \cdots]$.

1.2. If $A_1, A_2, \cdots, A_r$, $r \geqq n+3$, are distinct points of L_n, π: $A_1A_2 \cdots A_r$ denotes a *closed* polygon with the sides A_iA_{i+1}, $1 \leqq i \leqq r$, $(A_{r+s} = A_s)$ and vertices $A_1, A_2, \cdots, A_r$.

An (open) segment α: $A_iA_{i+1} \cdots A_{i+h}$, $0 \leqq h \leqq r-1$, is called an *arc* of π of length h.