PACIFIC JOURNAL OF MATHEMATICS
Vol. 39, No. 1, 1971

INFINITE MATRICES SUMMING EVERY ALMOST PERIODIC SEQUENCE

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Necessary and sufficient conditions are given for infinite matrices to sum every almost periodic sequence and their basic properties as summability matrices are studied. It is then shown that these matrices enter naturally in the problem of the determination of the jump or total quadratic jump of normalized functions of bounded variation on the circle in terms of the limits of matrix transforms of certain functions of their Fourier-Stieltjes coefficients. The results obtained generalize the classical theorems of Fejér and Wiener as also the extensions of theorems of Wiener given by Lozinskiĭ, Keogh, Petersen and Matveev. Applications are made to the study of coefficient properties of holomorphic functions in the unit disk with positive real part.

1. R. H. C. Newton [11] proved that a regular matrix $A = (a_{n,k})$ sums every periodic sequence if and only if $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} \exp(2\pi ikt)$ exists for each rational t. Vermes [15] generalized this result by proving that an arbitrary matrix $A = (a_{n,k})$ sums every periodic sequence if and only if for every rational t, (1) $\sum_{k=0}^{\infty} a_{n,k} \exp(2\pi ikt)$ converges and (2) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} \exp(2\pi ikt)$ exists.

The set P of all periodic sequences of complex numbers is a linear subspace of l_∞ that is not closed in the usual norm topology of the Banach space l_∞ since P is meager in l_∞. Berg and Wilansky [3] proved that the closure Q of P in l_∞ is the set of all semi-periodic sequences. (A sequence $x = \{x_k\}$ is called semi-periodic if for any $\varepsilon > 0$, there exists an integer r such that $|x_k - x_{k+rn}| < \varepsilon$ for every n and k). Berg [2], gave a characterization of infinite matrices summing every semi-periodic sequence which is rather involved. We first show that these matrices can be characterized simply as follows:

THEOREM 1. *An infinite matrix $A = (a_{n,k})$ sums every semi-periodic sequence if and only if* (1) $\|A\| = \sup_{n\geq 0} \sum_{k=0}^{\infty} |a_{n,k}| < \infty$ *and* (2) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} \exp(2\pi ikt)$ *exists for all rational* t.

Proof. If $x \in Q$, then for any $\varepsilon > 0$, there exists a $y \in P$ such that $\|x - y\|_\infty < \varepsilon$. If y is of period r, there exist constants $\lambda_1, \cdots, \lambda_r$ such that

$$\sum_{\nu=1}^{r} \exp(2\pi ik\nu/r) \cdot \lambda_\nu = y_k, \quad k = 0, 1, \cdots, r-1$$