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A GEOMETRIC APPROACH TO THE FIXED POINT INDEX

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J. Leray defined a local fixed point index for functions defined in what he called convexoid spaces. From the standpoint of analysis, the most important example of a convexoid space is a compact subset $C \subset X$, X a locally convex topological vector space, such that $C = \cup_{i=1}^{n} C_i$, where C_i are compact, convex subsets of X or a homeomorphic image of such a C. In this paper a simple geometric approach is given (see Lemma 2 below) by means of which a fixed point index can be defined for functions with domain in a class of spaces $\mathscr{F}$ which contains the spaces C mentioned above and also the compact metric ANR's. The usual properties of the fixed point index are established, and it is shown that they axiomatically determine the index for the class of spaces $\mathscr{F}$.

As far as we know, none of the approaches to the fixed point index which have been published since Leray's work have been shown to apply to spaces C of the type above. For instance, A. Granas [9] has remarked that if a compact space C is r-dominated by an open subset of an lctvs, then the Leray-Schauder index for compact maps on open subsets of an lctvs gives a fixed point index for maps of open subsets of C into C. But without metrizability the spaces considered here are not necessarily r-dominated by open subsets of an lctvs. F. Browder [3] has shown that if a compact Hausdorff space admits a "semicomplex structure," then a fixed point index can be defined for functions with domain in the space. However, to show that given topological spaces admit semicomplex structures, the metrizability of the spaces has almost invariable been used. Thus Browder has shown that compact, metric ANR's admit semicomplex structures, and Thompson [19] has established the same thing for metric HLC* spaces. But a finite union of compact, convex sets in an lctvs need not be metrizable. In any event, we shall avoid questions about semicomplex structure and obtain our fixed point index from the classical one for compact, finite dimensional polyhedra.

1. Let us begin with some notation. Let C be a compact subset of a locally convex topological vector space (lctvs) X, always assumed Hausdorff. We shall write $C \in \mathscr{F}_0$ if there exists a finite closed covering $\{C_i : 1 \leq i \leq n\}$ of C by compact, convex sets $C_i \subset C$, ie if $C = \bigcup_{i=1}^{n} C_i$, C_i a compact convex subset of X. If $C \in \mathscr{F}_0$, $G \subset C$