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GROUPS OF HOMEOMORPHISMS OF NORMED LINEAR SPACES

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For X a Hausdorff space let $H(X)$ be the group of homeomorphisms of X. We study here certain subgroups of $H(E)$ where E is an infinite-dimensional normed linear space.

The set of homeomorphisms from a topological space X onto itself forms a group $H(X)$ under composition. There are many topologies which can be given to $H(X)$, some of which may make $H(X)$ a topological group. It is natural to ask about the properties of $H(X)$, both algebraic and topological. Also, what relationships are there between X and $H(X)$? One way to attack these questions is to study various subgroups of $H(X)$. In this paper we shall investigate certain subgroups of $H(E)$, where E is a normed linear space.

1. Algebraic properties of $H(E)$. Let X be a Hausdorff space. If $A \subset X$, $S(A)$ will denote the set of elements of $H(X)$ which are supported on A. That is, $h \in S(A)$ if and only if $h|_{X-A}$ is the identity on $X-A$. Let $\mathscr{B}$ be a base for the topology on X. Define $B(X)$ to be the subgroup of $H(X)$ which is generated by those elements of $H(X)$ which are supported on elements of $\mathscr{B}$. Then $h \in B(X)$ if and only if $h = h_n \cdots h_1$, where for each $i \leqq n$, $h_i \in S(B_i)$ for some $B_i \in \mathscr{B}$. A homeomorphism $h \in H(X)$ is said to be stable if $h = h_n \cdots h_1$, where for each $i \leqq n$, $h_i \in S(X - U_i)$ for some nonempty open set U_i in X. The stable homeomorphisms of X, $SH(X)$, form a subgroup of $H(X)$.

We shall consider the following possible conditions on $\mathscr{B}$.

B 1. For every B_1, $B_2 \in \mathscr{B}$, there exists an $h \in H(X)$ such that $h(B_1) \subset B_2$.

B 1′. For every $B_1, B_2 \in \mathscr{B}$, there exists an $h \in B(X)$ such that $h(B_1) \subset B_2$.

B 2. For every $B \in \mathscr{B}$, there exists an $x \in B$ and a pairwise disjoint sequence $\{B_i \in \mathscr{B} \mid B_i \subset B, i = 1, 2, \cdots\}$ which converges to x (i.e., for every open set U containing x, there is some B_i contained in U), and there exists an $h \in S(B)$ such that $h(B_i) = B_{i+1}$ for every i.

B 3. For every $B \in \mathscr{B}$ and $h \in H(X)$, $h(B) \in \mathscr{B}$.

B 4. For every $B \in \mathscr{B}$, there exists $B' \in \mathscr{B}$ such that $B \cup B' = X$, and no $B \in \mathscr{B}$ is dense in X.