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A NOTE ON $C\theta\theta$-GROUPS

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A $C\theta\theta$-group is a finite group of order divisible by 3 in which centralisers of 3-elements are 3-groups. Several authors have studied such groups; in particular it is known that, given the additional hypothesis that the Sylow 3-subgroups intersect trivially, a simple $C\theta\theta$-group has abelian Sylow 3-subgroups. In this note it is proved that this additional hypothesis is superfluous.

More precisely the following will be proved:

THEOREM. *Let G be a $C\theta\theta$-group in which $0^3(G) = G$ and let M be a Sylow 3-subgroup of G. Then M is a TI-set in G.*

The proof of the theorem depends on two lemmas:

LEMMA 1. *Let H be a $C\theta\theta$-group. If any element of order 3 in H is conjugate to its inverse; or, equivalently, if any 3-local subgroup of H has even order; then Sylow 3-subgroups of H are abelian and hence TI-sets in H.*

Proof. Suppose t is an element of order 3 in H conjugate to its inverse. Let T be a Sylow 3-subgroup of H such that $t \in T$. Now the extended centraliser $C_H^*(t)$ is a Frobenius group with the 3-group $C_H(t)$ as kernel. Since $|C_H^*(t): C_H(t)| = 2$, $C_H(t)$ is abelian and every element in it is conjugate to its inverse. Now $Z(T) \leqslant C_H(t)$ so we may assume that $t \in Z(T)$. In this case $C_H(t) = T$ and so T is abelian.

LEMMA 2. *Let H be a $C\theta\theta$-group in which $0_3(H) > 1$. Then H is soluble and one of the following occurs:*
(i) *a Sylow 3-subgroup of H is normal in H*
(ii) $0^3(H) < H$.

Proof. Put $L = 0_3(H)$, $\bar{H} = H/L$. Suppose first that $|H|$ is even. Every element of L is conjugate to its inverse so, by Lemma 1, Sylow 3-subgroups of H are abelian. Clearly $L = C_H(L)$ is a Sylow 3-subgroup of H, case (i) arises, and $|\bar{H}|$ is prime to 3. $\bar{H}$ can now be regarded as a group of fixed-point-free automorphisms of L so, if p is odd, the Sylow p-subgroups of $\bar{H}$ are cyclic and the Sylow 2-subgroups are either cyclic or generalised quaternion. A group all