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FIXED POINTS AND STABILITY FOR A SUM OF TWO OPERATORS IN LOCALLY CONVEX SPACES

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Some fixed point theorems for a sum of two operators are proved, generalizing to locally convex spaces a fixed point theorem of M. A. Krasnoselskii, for a sum of a completely continuous and a contraction mapping, as well as some of its recent variants.

A notion of stability of solutions of nonlinear operator equations in linear topological spaces is formulated in terms of specific topologies on the set of nonlinear operators, and a theorem on the stability of fixed points of a sum of two operators is given. As a byproduct, sufficient conditions for a mapping to be open or to be onto are obtained.

1. Introduction. Several algebraic and topological settings in the theory and applications of nonlinear operator equations lead naturally to the investigation of fixed points of a sum of two nonlinear operators, or more generally, fixed points of a mapping on the Cartesian product $X \times X$ into X, where X is some appropriate space.

Fixed point theorems in topology and nonlinear functional analysis are usually based on certain properties (such as complete continuity, monotonicity, contractiveness, etc.) that the operator, considered as a single entity must satisfy. We recall for instance the Banach fixed point theorem, which asserts that a strict contraction on a complete metric space into itself has a unique fixed point, and the Schauder principle, which asserts that a continuous mapping F on a closed convex set K in a Hausdorff locally convex topological vector space X into K such that $F(K)$ is contained in a compact set, has a fixed point. In many problems of analysis, one encounters operators which may be split in the form $T = A + B$, where A is a contraction in some sense, and B is completely continuous, and T itself has neither of these properties. Thus neither the Schauder fixed point theorem nor the Banach fixed point theorem applies directly in this case, and it becomes desirable to develop fixed point theorems for such situations. An early theorem of this type was given by Krasnoselskii [12]: Let X be a Banach space, S be a bounded closed convex subset of X, and A, B be operators on S into X such that $Ax + By \in S$ for every pair $x, y \in S$. If A is a strict contraction and B is continuous and compact, then the equation $Ax + Bx = x$ has a solution in S. The proof of this theorem is quite simple, given the Schauder principle.

Krasnoselskii's theorem is an example of an algebraic setting which