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ON SATURATED REDUCED PRODUCTS

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The first part of this paper characterizes the filters whose reduced products are saturated with respect to quantifier free formulas. It is shown that filters with this property are exactly good filters whose Boolean algebra is compact. In the second part we investigate their set-theoretical properties and prove that such filters exist.

This paper contains a proof of the result announced in [1] as well as a number of results directly connected with this theorem.

The problem is to characterize those filters which have the property that products reduced by these filters are κ-saturated. The special case of the problem in which we talk about ultrafilters only was completely solved by Keisler in [6]. The above problem was first attacked by F. Galvin and also by Ph. Olin and B. Jónsson (see [5]). They showed that the filter of cofinite subsets over ω is ω_1-saturative. Galvin's results were a little more general. In [11], L. Pacholski and C. Ryll-Nardzewski described those atomless filters which were ω_1-saturative. The author of this article then obtained a characterization of (Φ_0, κ)-saturative filters for any κ. S. Shelah after reading a sketch of this article, obtained, using a different method (see [3]), a characterization of κ-saturative filters, thus solving the problem. (For $\kappa = \omega$, it was independently solved also by L. Pacholski.)

The first part of this paper deals with (Φ_0, κ)-saturatedness of reduced products. It is believed that theorems proved here have some applications to algebra. In possible applications the notion (Φ_0, κ)-saturatedness seems, better suited than the full saturatedness ([10]) because e.g., a solution of a system of equations is expressible by quantifier-free formulas. In the second part we deal with products of different kind of filters and we apply these results to get an existence theorem for excellent filters. The third part is devoted to discussion and open problems.

Our notation is standard. λ, κ stand always for cardinals. 0 is the empty set as well as the least element in Boolean algebras. $S_\omega(X)$ is the set of finite subsets of X. If f is a function from X into Y we write it sometimes as $f: X \to Y$. If $Z \subseteq X$ then $f^*(Z) = \{f(z) \mid z \in Z\}$ and $f \mid Z$ is the function f restricted to Z. A subset X of a Boolean algebra is said to have the finite intersection property if for any $x_1, \cdots, x_n \in X$ $x_1 \cap \cdots \cap x_n \neq 0$. We write it often as $FIP(X)$. D usually stands for a filter over the set $I = \cup D$. The ideal $\{x \subseteq I \mid I - x \in D\}$