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REAL-VALUED CHARACTERS OF METACYCLIC GROUPS

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The nonlinear real-valued irreducible characters of metacyclic groups are determined, and the defining relations are given for the metacyclic groups with every nonlinear irreducible character real-valued.

Consider the metacyclic group

$$G = \langle a, b \mid a^n = b^m = 1, a^k = b^t, b^{-1}ab = a^r \rangle$$

where $r^t - 1 \equiv kr - k \equiv 0 \pmod{n}$ and $t \mid m$. Let s be a positive divisor of n and let t_s be the smallest positive integer such that $r^{t_s} \equiv 1 \pmod{s}$. Let χ_s be a linear character of $\langle a \rangle$ with kernel $\langle a^s \rangle$ and $\bar{\chi}_s$ be an extension of χ_s to $K_s = \langle a, b^{t_s} \rangle$, see [1]. From [1] the induced character $\bar{\chi}_s^G$ is irreducible of degree t_s and every irreducible character of G is some $\bar{\chi}_s^G$.

Assume $\bar{\chi}_s^G$ is nonlinear. Then $K_s \subset G \neq K_s$. From Lemma 1 of [2], $\bar{\chi}_s^G$ is real-valued if and only if there is $y \in G$ such that $\langle K_s, y \rangle / D_s$ is dihedral or quaternion, where D_s is the kernel of $\bar{\chi}_s$. Assume such a y exists. Since G/K_s is cyclic, t_s is even and we may let $y = b^{t_s/2}$. Hence $r^{t_s/2} \equiv -1 \pmod{s}$ and $\bar{\chi}_s(b^{t_s}) = \pm 1$. Since $\bar{\chi}_s(b^{t_s})^{t/t_s} = \chi_s(a^k)$, $\bar{\chi}_s(b^{t_s}) = \pm 1$ implies either (i) $s \mid (k, n)$, or (ii) $s \mid 2(k, n)$, $s \nmid (k, n)$, and $t = t_s$. When (i) occurs, $\bar{\chi}_s(b^{t_s}) = 1$ if t/t_s is odd and $\bar{\chi}_s(b^{t_s}) = \pm 1$ if t/t_s is even. When (ii) occurs $\bar{\chi}_s(b^{t_s}) = -1$. Note that if $\bar{\chi}_s(b^{t_s}) = -1$ then $\bar{\chi}_s^G$ is not realizable in the real field. Using [1] the number of the nonlinear irreducible real-valued characters not realizable in the real field is $\Sigma'' \phi(s)/t_s$ where Σ'' is over all positive divisors s of n such that t_s is even, $r^{t_s/2} \equiv -1 \pmod{s}$ and either (i) $s \mid (k, n)$ and t/t_s even or (ii) $s \mid 2(k, n)$, $s \nmid (k, n)$, and $t = t_s$. The number of the nonlinear irreducible real-valued characters is $\Sigma' \phi(s) | t_s + \Sigma'' \phi(s)/t_s$ where Σ' is defined in [2, § 2].

Let $\pi = \{p \mid p \text{ an odd prime dividing } n\}$.

THEOREM. *Assume G, given as above, is non-abelian. Then every nonlinear irreducible character of G is real-valued if and only if n, m, k, t, t_p and r satisfy one of the conditions below.*

(a) $m = t$ *with either* (i) $4 \nmid n$, $2 \mid t$, *and* $t_p = t$ *for all* $p \in \pi$, (ii) $4 \nmid n$, $4 \mid t$, *and* $t_p = t/2$ *for all* $p \in \pi$, *or* (iii) $4 \mid n$, $r = -1$, $t = 2$ *or* $t = 4$.

(b) $m = 2t$ *with either* (i) $2 \| n$, $2 \mid t$, *and* $t_p = t$ *for all* $p \in \pi$, *or*