PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 1, 1972

LATTICES OF RADICALS

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Nothing first that the class of all radicals for associative rings forms a lattice under a natural ordering, we show that several important subclasses, including the class of hereditary radicals, form sublattices. We give an example showing that the special radicals do not form a sublattice even though they form a complete lattice under the same ordering.

After we have studied various lattice-theoretic properties of our main lattices, showing, in particular, that the lattice of hereditary radicals is Brouwerian, we determine the atoms of that lattice and show that it has no dual atoms by computations with free rings. We characterize pseudocomplements in the lattice of hereditary radicals and give partial results toward determining which radicals of that lattice are complemented.

We note that W. G. Leavitt [13] has remarked on some of the properties of meet and join radicals in the category of non-associative rings. We work in the category of all associative rings because of the pathology of radical theory in more general categories. (See for example [5].)

For definitions and elementary properties of radicals, see [8] or [12]. The latters α, β, $\cdots$ will denote radicals, λ, μ, $\cdots$ ordinals.

1. The Lattice of All Radicals.

The results of this section are essentially a reformulation of Leavitt's work in lattice-theoretic language.

The collection of all radicals can be partially ordered by defining $\alpha \leqq \beta$ if $\alpha(R) \subseteqq \beta(R)$ for all rings R. This is equivalent to the statement that all α-radical rings are β-radical, or that all β-semisimple rings are α-semisimple.

Let $\mathscr{R}_\alpha$ denote the class of all α-radical rings, and $\mathscr{S}_\alpha$ the class of all α-semisimple rings. If $\mathscr{R}$ is a radical class, let $\alpha_{\mathscr{R}}$ denote the radical associated with $\mathscr{R}$, and if $\mathscr{S}$ is a semisimple class, let α denote the radical associated with $\mathscr{S}$.

PROPOSITION 1. *The class of all radicals forms a complete lattice, where for any collection $\{\alpha_i\}_{i \in I}$ of radicals, the semisimple class of the join is $\bigcap_i \mathscr{S}_{\alpha_i}$ and the radical class of the meet is $\bigcap_i \mathscr{R}_{\alpha_i}$.*

Proof. Leavitt [13] has shown that $\bigcap_i \mathscr{S}_{\alpha_i}$ is a semisimple class and $\bigcap_i \mathscr{R}_{\alpha_i}$ is a radical class. They clearly are the join and meet