PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No 1, 1972

PROJECTIVE LATTICES AND BOUNDED HOMOMORPHISMS

ALAN KOSTINSKY

The main purpose of this paper is to prove that a finitely generated lattice is projective iff it is imbeddable in a free lattice. This result appears as a consequence of a more general theorem, in which a sufficient condition for projectivity is given in terms of the notion (due to Ralph McKenzie) of bounded homomorphism.

In [1, Theorems 4.1, 4.4] Baker and Hales completely describe the distributive projective lattices and obtain as a corollary the fact that a finite distributive lattice is projective iff it is imbeddable in a free lattice. This last result has been improved by McKenzie, who finds in [6, proof of Theorem 6.3] that for any finite lattice L, L is projective iff it is imbeddable in a free lattice. McKenzie's proof uses some ideas due to B. Jónsson. To extend the theorem to finitely generated lattices we sharpen arguments of [6]. As stated above, we use the notion of bounded homomorphism; this idea is defined by McKenzie in [6], and it plays an important role in that paper. Theorem 3.4 below was first announced in the author's abstract [4].

1. Preliminaries. We regard a lattice as an algebraic structure $\langle L, +, \cdot\rangle$ in which the sum (join) and product (meet) satisfy the usual equational axioms. It will not cause confusion to refer to a lattice by naming its universe. We denote by $\leqq$ the ordering of the lattice L, that is, the partial ordering naturally associated with L ($x < y$ means $x \leqq y$ and $x \neq y$). If the greatest lower bound (least upper bound) of a subset U of L exists in L, it is denoted $\bigwedge U$ ($\bigwedge U$).

The notation and terminology used for maps is largely standard. By an epimorphism of a lattice L into a lattice M we mean a homomorphism of L onto M. For any sets L, M, N, and any maps $f\colon L \rightarrow M$ and $g\colon M \rightarrow N$, the composite map (of L into N) is denoted $g \circ f$. (In the arrow notation, maps—whether homomorphisms or not—which are onto may be indicated by the use of a double-headed arrow, $\twoheadrightarrow$.)

A chain is a lattice whose ordering is a linear ordering. A chain is bounded iff it has a least element and a greatest element. Any ordinal α may be viewed as the chain whose ordering is the natural ordering of α. The set of all natural numbers is denoted ω.

We regard Boolean algebras as lattices (thus, zero, one, and complementation are not primitive operations).