PACIFIC JOURNAL OF MATHEMATICS
Vol. 40, No. 1, 1972

EVERY GENERALIZED PETERSEN GRAPH HAS A TAIT COLORING

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Watkins has defined a family of graphs which he calls generalized Petersen graphs. He conjectures that all but the original Petersen graph have a Tait coloring, and proves the conjecture for a large number of these graphs. In this paper it is shown that the conjecture is indeed true.

DEFINITIONS. Let n and k be positive integers, $k \leqq n-1$, $n \neq 2k$. The generalized Petersen graph $G(n, k)$ has $2n$ vertices, denoted by $\{0, 1, 2, \cdots, n-1; 0', 1', 2', \cdots, \cdots, (n-1)'\}$ and all edges of the form $(i, i+1)$, (i, i'), $(i', (i+k)')$ for $0 \leqq i \leqq n-1$, where all numbers are read modulo n. $G(5, 2)$ is the Petersen graph. See Watkins [2].

The sets of edges $\{(i, i+1)\}$ and $\{(i', (i+k)')\}$ are called the outer and inner rims respectively and the edges (i, i') are called the spokes.

A Tait coloring of a trivalent graph is an edge-coloring in three colors such that each color is incident to each vertex. A 2-factor of a graph is a bivalent spanning subgraph. A 2-factor consists of disjoint circuits. A Tait cycle of a trivalent graph is a 2-factor all of whose circuits have even length. A Tait cycle induces a Tait coloring and conversely.

The method that Watkins used in proving that many generalized Petersen graphs have a Tait coloring was to prove that certain color patterns on the spokes induce a Tait coloring. Our method for the remaining cases consists of the construction of 2-factors and of proof that these 2-factors are Tait cycles under appropriate conditions.

We restrict ourselves to the generalized Petersen graphs $G(n, k)$ with the properties:

$$n \text{ odd}, \; n \geqq 7, \; (n, k) = 1, \text{ and } 2 < k < \frac{n-1}{2} .$$

All other cases (and some special instances of the above) were dealt with by Watkins.

We construct three types of 2-factors. The first type is a Tait cycle when k is odd. The second type is a Tait cycle when k is even and $n \equiv 3 (\text{mod } 4)$ and also when k is even and $n \equiv 1 (\text{mod } 4)$ with k^{-1} even. (As $(n, k) = 1$, we define k^{-1} as the unique positive integer $< n$, for which $kk^{-1} \equiv 1 (\text{mod } n)$.) The third type takes care of the remaining graphs.