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THE OPEN MAPPING THEOREM FOR SPACES WITH UNIQUE SEGMENTS

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If X and Y are spaces with unique segments, an affine map from X into Y is a map which takes segments into segments. The purpose of this paper is to give conditions on spaces X and Y such that we can prove the following versions of the Open Mapping and Closed Graph Theorems: (1) a continuous affine map of X onto Y is open, and (2) an affine map of X onto Y with closed graph is continuous.

Let X and Y be spaces with unique segments, with $\Phi(x, y, t)$ $(0 \leqq t \leqq 1)$ denoting the intrinsically parametrized segment from x to y. A map T between two such spaces is said to be affine if $T(\Phi(x, y, t)) = \Phi(Tx, Ty, t)$, and a subset A is convex if $x, y \in A$, $0 \leqq t \leqq 1 \Rightarrow \Phi(x, y, t) \in A$. The purpose of this paper is to prove versions of the Open Mapping and Closed Graph theorems for classes of spaces with unique segments.

Throughout this paper, all metric spaces will be spaces with unique segments (unique curves of minimal, realizing the distance, length between any two points). The open sphere with center x and radius ε will be denoted by $S(x, \varepsilon)$.

DEFINITION 1. (X, d) is said to be regular if it is complete, the closure of convex sets is convex, open spheres are convex, and $x_n \rightarrow x_0 \Rightarrow \Phi(z, x_n, \alpha) \rightarrow \Phi(z, x_0, \alpha)$ for $z \in X$, $\alpha \in [0, 1]$.

DEFINITION 2. A sphere $S(x_0, \varepsilon)$ is said to be thick if for any $y \in S(x_0, \varepsilon)$ and $x \in X$, $x \neq y$, there is a $z \in S(x_0, \varepsilon)$ and $\alpha \in (0, 1]$ such that $y = \Phi(z, x, \alpha)$.

It is always possible to extend geodesics into thick spheres.

DEFINITION 3. A sphere $S(x_0, \varepsilon)$ is said to be round if, given $x \in S(x_0, \varepsilon)$, $y \in S(x, \varepsilon - d(x_0, x))$, and λ such that $d(x, y) < \lambda < \varepsilon - d(x_0, x) \Rightarrow$ there is a $z \in S(x, \varepsilon - d(x_0, x))$ and $\alpha \in (0, 1)$ such that $d(x, z) = \lambda$, $y = \Phi(x, z, \alpha)$.

Given any sphere $S(x, \delta)$ contained in a round sphere $S(x_0, \varepsilon)$ and any $y \in S(x, \delta)$, y lies on a geodesic connecting x and a point in $S(x, \delta)$ whose distance from x is arbitrarily close to δ. It should be noted that both thickness and roundness are hereditary properties; that is, if $S(x_0, \varepsilon)$ is thick (round) and $S(x, \delta) \subseteqq S(x_0, \varepsilon)$ then $S(x, \delta)$ is thick (round).

We now prove an Open Mapping theorem.