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DISJOINT MAXIMAL INVARIANT SUBSPACES

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THEOREM. **If there exists a maximal invariant subspace of $H^2_{\mathscr{H}}$ not of codimension 1, then there exists an uncountable family $\{\mathscr{M}_\alpha\}$ of maximal invariant subspaces such that $\mathscr{M}_\alpha \cap \mathscr{M}_\beta = (0)$ if $\alpha \neq \beta$.**

$H^2_{\mathscr{H}}$ is the (separable) Hilbert space of all functions $F(e^{i\theta})$ defined on the unit circle with values in the separable *infinite dimensional* Hilbert space $\mathscr{H}$, and which are weakly in the Hardy class H^2. For a closed subspace of $H^2_{\mathscr{H}}$, "invariant" means invariant under the right shift operator. In [5], the existence of an uncountable family of pairwise disjoint full range invariant subspaces was established, and it was remarked that while the theorem said full range invariant subspaces could be small in this sense, there was reason to believe maximal invariant subspaces (if they exist) would exhibit the same behavior. We now prove this, making essential use of a theorem of Dixmier on operator ranges. The author is grateful to Jim Williams for interpreting Dixmier's results to him.

The existence of a maximal invariant subspace of $H^2_{\mathscr{H}}$ not of codimension 1 is equivalent to the existence of a bounded operator on $\mathscr{H}$ with no nontrivial invariant subspaces in the usual sense [3, p. 103]. Bounded one-to-one operators on $\mathscr{H}$ with disjoint ranges have disjoint Rota subspaces [4, p. 169], and these will be maximal if the operators have no invariant subspaces. It therefore suffices to show, assuming the existence of one bounded operator without invariant subspaces, that there are uncountably many such operators with disjoint ranges. If T has no invariant subspaces, then T has purely continuous spectrum, and since T and $(T - \lambda I)$ have the same invariant subspaces, we can assume the range of T (obviously dense) is not all of $\mathscr{H}$. It follows that the codimension of the range of T is in fact uncountably infinite [2, Theorem 3.6, Corollary 1]. Dixmier's theorem [1, p. 84] asserts there is a continuous unitary group $\{U_t\}$, $-\infty < t < \infty$, such that $U_t T U_t^{-1}$ and $U_s T U_s^{-1}$ have disjoint ranges for $s \neq t$. The proof of Theorem 3.6 of [2] can be generalized to yield uncountably many disjoint operator ranges by choosing instead of the function V, a family V_t defined as multiplication by 1 on $[0, t]$, and multiplication by -1 on $[t, 2\pi]$.