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UNIFORM FINITE GENERATION OF THE AFFINE GROUP

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A connected Lie group H is said to be uniformly finitely generated by a given pair of one-parameter subgroups if there exists a positive integer n such that every element of H can be written as a finite product of length at most n of elements chosen alternately from the two one-parameter subgroups. Define the order of generation of H as the least such n. It is shown that the order of generation of the affine group is either 4 or 5 while its connected Lie subgroups (with two exceptions) have order of generation equal to their dimension.

A connected Lie group H is generated by a pair of one-parameter subgroups if every element of H can be written as a finite product of elements chosen alternately from the two one-parameter subgroups. If, moreover, there exists a positive integer n such that every element of H possesses such a representation of length at most n, then H is said to be uniformly finitely generated by the pair of one-parameter subgroups. In this case define the order of generation of H as the least such n; otherwise define it as infinity. Since the order of generation of H will, in general, depend upon the pair of one-parameter subgroups, H may have many different orders of generation. However, it is a simple consequence of Sard's theorem [4] that the order of generation of H must always be greater than or equal to the dimension of H.

The orders of generation of the isometry groups of the Euclidean and Non-Euclidean geometry are known. The order of generation of the isometry group of the spherical geometry may be any integer $\geqq 3$; it is determined by the cross-ratio of the fixed points of the pair of elliptic one-parameter subgroups [1]. The order of generation of the isometry group of the Euclidean geometry is infinite if both one-parameter subgroups are elliptic and it is 3 if one is elliptic and the other parabolic [2]. The order of generation of the isometry group of the hyperbolic geometry is finite if both one-parameter subgroups are elliptic, 3 if exactly one is elliptic and 4 in all other cases except that it is 6 if both are hyperbolic with interlacing fixed points [2].

Here all possible orders of generation of the affine group, i.e., the group of all transformations $w = \alpha z + \beta$ (α, β complex, $\alpha \neq 0$) and of all its connected Lie subgroups are determined. It is shown that for the affine group the possible orders of generation are 4 and 5 while its connected Lie subgroups (excluding the isometry group of the Euclidean geometry and the group $w = az + \beta$, $a > 0$) have order of