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# ON THE SPECTRAL RADIUS FORMULA IN BANACH ALGEBRAS

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**$B$ will always denote a commutative semi-simple Banach algebra with a unit element. If $f \in B$ then $r(f)$ denotes its spectral radius. A sequence $F = (f_j)_1^\infty$ is called a spectral null sequence if $\|f_j\| \leqq 1$ for each $j$, while $\lim_{j\to\infty} r(f_j) = 0$. If $F = (f_j)$ is a spectral null sequence we put $r_N(F) = \limsup_{j\to\infty} \|f_j^N\|^{1/N}$ for each $N \geqq 1$. Finally we define the complex number $r_N(B) = \sup\{r_N(F): F$ is a spectral null sequence in $B\}$. In general $r_N(B) = 1$ for all $N \geqq 1$ and the aim of this paper is to study the case when $r_N(B) < 1$ for some $N$.**

We say that $B$ satisfies a bounded inverse formula if there exists some $0 < \varepsilon < 1$ and a constant $K_0$ such that for all $f$ in $B$ satisfying $\|f\| \leqq 1$ and $r(f) \leqq \varepsilon$, it follows that $\|(e - f)^{-1}\| \leqq K_0$. In Theorem 3.1. we prove that $B$ satisfies a bounded inverse formula if and only if $r_N(B) < 1$ for some $N$.

In § 1 we give a criterion which implies that $B$ is a sup-norm algebra. In § 2 we introduce the so called infinite product of $B$ which will enable us to study spectral null sequences in § 3.

**1. Sup-norm algebras.** Recall that $B$ is a sup-norm algebra if there exists a constant $K$ such that $\|f\| \leqq Kr(f)$ for all $f$ in $B$. Clearly this happens if and only if $r_1(B) = 0$. Next we give an example where $r_1(B) = 1$ while $r_2(B) = 0$.

Let $B = C^1[0, 1]$ be the algebra of all continuously differentiable functions on the closed unit interval. If $f \in B$ we put $\|f\| = \sup\{|f(x)| + |f'(y)|: 0 \leqq x, y \leqq 1\}$. The maximal ideal space $M_B$ can be identified with $[0, 1]$, so the spectral radius formula shows that $r(f) = \sup\{|f(x)|: 0 \leqq x \leqq 1\}$. From this we easily deduce that $r_2(B) = 0$. In fact we also notice that $\|f^n\| \leqq n\|f\|(r(f))^{n-1}$ holds for all $n \geqq 2$. We will now prove that this estimate is sharp.

THEOREM 1.1. *Let the norm in $B$ satisfy $\|f^n\| \leqq qn\|f\|r(f)^{n-1}$ for some $q < 1$ and some $n \geqq 2$. Then $B$ is a sup-norm algebra and there is a constant $K(n, q)$ such that $\|f\| \leqq K(n, q)r(f)$ for all $f \in B$.*

LEMMA 1.2. *Let $n \geqq 3$ and suppose that $\|f^n\| \leqq K\|f\|r(f)^{n-1}$ for all $f$ in $B$ and some constant $K$. Then there is a constant $K(n)$ such that $\|f^2\| \leqq K(n)K\|f\|r(f)$.*