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INJECTIVE MODULES OVER DUO RINGS

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Let R be a ring with unit whose right and left ideals are two-sided ideals. It is shown that every Noetherian injective R-module has finite length (i.e., has a finite composition series). If I is a maximal ideal of R, then R has a universal localization, R_I at I. The condition that the injective hull of R/I is finite is characterized in terms of R_I.

1. **Introduction.** In this note all rings have unit and modules are unital right modules. A. Rosenberg and D. Zelinsky have shown that if R is a commutative ring and I is a maximal ideal of R, then the injective hull of R/I is finite (i.e., has finite length) if and only if the localization of R at I is Artinian (see Theorem 5 of [6, p. 379]). In this note we shall prove an extended version (Theorem 4) of their result for a class of rings which is somewhat interesting in itself. Let us call a ring R a *duo ring* if $xR = Rx$ for all $x \in R$ (equivalently all ideals are bilateral). Such rings were investigated by E. Feller [2] and G. Thierrin [7]. Trivial examples of duo rings are, of course, commutative rings and division rings. Nontrivial duo rings are not difficult to come by (e.g., any noncommutative special primary ring is duo, since the only right or left ideals are powers of the unique maximal ideal). In fact some interesting examples of duo rings have already occurred in the literature: M. Auslander and O. Goldman have shown in [1, p. 13] that there exist noncommutative maximal orders which are both duo rings and Noetherian domains. Further investigations of such rings have been carried out by G. Maury in [4].

One of the basic difficulties in extending Rosenberg and Zelinsky's result to duo rings is the existence of suitable localizations. This problem is considered in §2. Next we show in §3 that Noetherian duo rings are classical in the sense that the familiar primary decomposition theory of commutative Noetherian rings extends to duo rings. We use this fact to show that Noetherian injective modules over duo rings are finite. Finally we prove our main result in §4.

The injective hull of the module M will be denoted by $E(M)$. If A and B are subsets of M or R, then $A \,.\!\cdot\, B = \{x \in R \mid xB \subseteqq A\}$ and $A \,\cdot\!.\, B = \{x \in R \mid Ax \subseteqq B\}$. Also $A \backslash B$ is the set of elements in A but not B.

2. **Localizations.** First of all we need a suitable definition of the term "localization." The ideal P of R is prime if $AB \subseteqq P$ implies $A \subseteqq P$ or $B \subseteqq P$ for all ideals A and B of R.