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SOME GEOMETRIC PROPERTIES RELATED TO THE FIXED POINT THEORY FOR NONEXPANSIVE MAPPINGS

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The main result of this paper asserts that if a Banach space admits a sequentially weakly continuous duality function, then a condition introduced by Opial to characterize weak limits by means of the norm is satisfied and the space has normal structure in the sense of Brodskii-Milman. This result of geometric nature allows some unification in the fixed point theory for both single-valued and multi-valued non-expansive mappings.

Let K be a nonempty weakly compact convex subset of a real Banach space X and let T be a nonexpansive mapping of K into its nonempty compact subsets (i.e., $D(Tx, Ty) \leqq \|x - y\|$ for all $x, y \in K$, where $D(,)$ denotes the Hausdorff metric). While the question of the existence of a fixed point for T remains open, several positive results were proved recently under various conditions of geometric type on the norm of X. We list here the conditions we have in mind:

(I) (Browder [5]) X admits a sequentially continuous duality function $F_\phi: X, \sigma(X, X^*) \to X^*, \sigma(X^*, X)$ (i.e., a function F_ϕ such that $\langle x, F_\phi(x)\rangle = \|x\| \|F_\phi(x)\|$ and $\|F_\phi(x)\| = \phi(\|x\|)$ for all $x \in X$, where $\phi: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ is continuous strictly increasing with $\phi(0) = 0$ and $\phi(+\infty) = +\infty$).

(II) (Opial [17]) If a sequence $\{x_n\}$ converges weakly in X to x_0, then $\liminf \|x_n - x\| > \liminf \|x_n - x_0\|$ for all $x \neq x_0$.

(III) (Brodskii-Milman [4]) Every weakly compact convex subset H of X has normal structure (i.e., for each convex subset L of H which contains more than one point there exists $x \in L$ such that $\sup\{\|x - y\|; y \in L\} < \sup\{\|u - v\|; u, v \in L\}$).

When T is single-valued, the existence of a fixed point for T in K was proved by Browder [5] if X satisfies (I) and if T can be extended outside K in a nonexpansive way, and by Kirk [12] if X satisfies (III). A similar situation occurs in the multivalued case where one also encounters two different approaches: one by Browder [6] who proved a fixed point theorem under condition (I) and some additional assumptions, and another by the second author [14] who obtained the same conclusion under condition (II).

It is a consequence of our main theorem that in both cases the