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SOME RESULTS ON NEAREST POINTS AND SUPPORT PROPERTIES OF CONVEX SETS IN c_0

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The space c_0 is shown to contain a closed and bounded symmetric convex body such that no point of its complement has a nearest point in it. Related results involving the existence of functionals which support each member of a family of convex sets are also discussed.

1. Introduction and preliminary results. It has been shown in [3] that if X is a separable conjugate Banach space (i.e., if $X = E^*$ where E is a normed linear space and X contains a countable dense set) and if S is a closed, bounded set in X then, for every nonnegative real number d, there exist x in X and s_0 in S such that

$$d = \|x - s_0\| = \inf\{\|x - s\|: s \in S\} .$$

Further, it was shown that under the additional assumptions that the unit ball in X and the weak* closed convex hull of S are both strictly convex, the set of points in X admitting nearest points in S is weak* dense in X. The aim of the present paper is to define more precisely the relationship between these geometrical properties and the assumption that X is a separable conjugate space. The paper is concerned, for the most part, with the behaviour of c_0 in this respect. As is well known, this space is separable but not a conjugate space.

Our results show, first of all, that c_0 belongs to the class N_2 ([4]), i.e., the class of those Banach spaces which contain a closed, bounded convex set such that no point in its complement has a nearest point in the set; thus correcting an oversight of Klee. In the third section extensions of this result are presented in two directions. Finally, it is shown that, in a certain sense, the geometry of c_0 on the one hand and that of separable conjugate spaces on the other, are diametrically opposed; here we are indebted to V. L. Klee for remarks (in a private communication) which led us in this direction.

We have tried to obtain results (one way or the other) about m — a conjugate, nonseparable space—but have so far failed.

Before coming to the main theorem we give a preliminary proposition which relates various geometric properties.

PROPOSITION 1. *Let X be a real normed linear space and C a closed, bounded, convex set in X. Let N denote the set of points in $X \backslash C$ which have a nearest point in C. Then the following are equivalent:*