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# OPERATOR-VALUED INNER FUNCTIONS ANALYTIC ON THE CLOSED DISC

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**It is shown that the class of operator valued inner functions analytic on the closed disc is sufficiently large for the invariant subspace problem. These inner functions are then transferred to the upper half-plane and studied via the differential equation $U' = iAU$. The relationship between $A$ and $U$ is investigated. Necessary and sufficient conditions are given on $A$ for $U$ to be the Potapov inner function of a normal operator.**

1. First we establish our notation. $H$ is a fixed separable, infinite dimensional Hilbert space. $H_H^2$ is the usual Hardy spaces of the circle or upper half-plane with values in $H$. Which $H_H^2$ will be clear from the context. We will always denote a real variable by $x$, variables in the disc by $u$ or $w$, and variables in the upper half-plane by $z$. $S$ denotes multiplication by $w$. $S^*$ is the adjoint of $S$ restricted to $H_H^2$. An operator-valued function $U(w)$, defined on the circle, is inner if $Uf \in H_H^2$ for every $f \in H_H^2$ and $U(w)$, $|w| = 1$, is unitary. $U$ has an analytic extension into the unit disc which we shall identify with $U$. The map $z = i(1-w)/(1+w)$, $w = (z-i)/(z+i)$, transfers $U(w)$ to $U(z)$, an inner function on $H_H^2$ of the upper half-plane. $U'$ will always denote differentiation with respect to $z$, or $x$ if we are restricting $U$ to the real axis. If $T$ is a bounded operator on $H$, $r(T)$ is its spectral radius, $N(T)$ is its null space. If $M$, $R$ are subspaces of a Hilbert space, then $M \ominus R = M \cap R^\perp$ and $\oplus$ is an orthogonal sum.

2. If $T$ is an operator on $H$, $\|T\| \leqq 1$, and $T^n \to 0$ strongly, then associated to $T$ is an inner function $V_T(w)$ called the Potapov inner function for $T$. If $K = H_H^2 \ominus V_T H_H^2$, then $S^*$ restricted to $K$ is similar to $T$. The question then of invariant subspaces for $T$ reduces to finding invariant subspaces for $S^*$ restricted to $K$. This is equivalent to whether $V_T$ factors into two nonconstant inner functions. For a more complete discussion of these ideas see [6, 8].

3. Two special classes already proposed are the $(IN)$-operators of Herrero [7] and the scalar inner operators of Sherman [11]. Both of these classes are too restrictive for the invariant subspace problem. Sherman's calculations can be made to show $V_T$ is scalar if and only if $T$ is normal. In this case the invariant subspace problem