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ARITHMETICAL PROPERTIES OF GENERALIZED RAMANUJAN SUMS

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The sums studied in this paper are defined as follows. For any two arithmetical functions f and g, let

$$S_{f,g}(m, k) = \sum_{d \mid (m,k)} f(d)g(k/d) , \tag{1}$$

where the sum extends over the divisors of the greatest common divisor (m, k) of the positive integers m and k. It should be noted that m and k do not enter symmetrically in (1) unless g is constant.

The sums $S_{f,g}(m, k)$ generalize the Dirichlet convolution

$$(f*g)(k) = \sum_{d \mid k} f(d)g(k/d) , \tag{2}$$

to which they reduce when $(m, k) = k$. Multiplicative properties and finite Fourier expansions were obtained in [1]. A famous special case is Ramanujan's sum $c_k(m)$, the sum of the mth powers of the primitive kth roots of unity, for which we have

$$c_k(m) = \sum_{\substack{h \bmod k \\ (h,k)=1}} \exp(2\pi imh/k) = \sum_{d \mid (m,k)} d\mu(k/d) , \tag{3}$$

where μ is the Möbius function. The second sum in (3) is an example of (1) with $f(n) = n$ and $g(n) = \mu(n)$ for all n. When $(m, k) = 1$ we have $c_k(m) = \mu(k)$, and when $(m, k) = k$ we have $c_k(m) = \varphi(m)$, Euler's totient.

In a study on cyclotomic polynomials, Hölder [4] showed that Ramanujan's sum can also be expressed in closed form as follows:

$$c_k(m) = \frac{\varphi(m)}{\varphi(m/(k, m))}\mu(m/(k, m)) . \tag{4}$$

The number on the right is called the Von Sterneck function and is denoted by $\Phi(m, k)$. Thus, (4) states that

$$c_k(m) = \Phi(m, k) .$$

The function $\Phi(m, k)$ was encountered by Von Sterneck in 1902 [11] in a study of restricted partitions with summands reduced to their least residues module m. Its properties were also studied by Nicol and Vandiver [7].

We derive further properties of the sums $S_{f,g}(m, k)$. Some of them