PACIFIC JOURNAL OF MATHEMATICS
Vol. 41, No. 3, 1972

FIXED POINT THEOREMS FOR NONEXPANSIVE MAPPINGS

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The notions of nonexpansive, contractive, iteratively contractive and strictly contractive mappings have been generalized to a Hausdorff topological space whose topology is generated by a family of pseudometrics. A fixed point theorem for strictly contractive mappings is obtained which generalizes the Banach's contractive mapping principle. Several examples and an implicit function theorem are given as well as some applications in solving functional equations in topological vector spaces.

For iteratively contractive mappings, some results obtained by D. D. Ang and E. D. Daykin, S. C. Chu and J. B. Diaz, by M. Edelstein, by K. W. Ng and by E. Rakotch respectively are generalized.

1. Definitions and Notations.

Throughout this paper X is a Hausdorff topological space whose topology is generated by a family $\{d_\lambda\}_{\lambda\in\Gamma}$ of pseudometrics on X. It is well known that in order for X to be such a space, it is necessary and sufficient that X be a Hausdorff uniform space, or equivalently a Hausdorff completely regular space. It is clear that for any $x, y \in X$, if $x \neq y$, then there is an $\lambda \in \Gamma$ such that $d_\lambda(x, y) > 0$. We shall denote by $\mathfrak{J}^+$ the set of all nonnegative integers, $\mathfrak{N}$ the set of all natural numbers, $\mathfrak{R}$ the set of real numbers and $\mathfrak{C}$ the set of all complex numbers.

NOTATION 1.1. If $f, g: X \to X$, we shall denote by fg the composition $f\circ g$ of f and g. If $n \in \mathfrak{J}^+$, we shall denote $f^{n+1} = f^n(f)$, where $f^0 = I$, the identity mapping of X.

NOTATION 1.2. If $A \subset X$ is nonempty, for each $\lambda\in\Gamma$, we denote $d_\lambda(A) = \sup\{d_\lambda(x, y): x, y \in A\}$, which is called the diameter of A w.r.t. d_λ.

DEFINITION 1.3. If $f: X\to X$, then

(i) f is nonexpansive w.r.t. $\{d_\lambda\}_{\lambda\in\Gamma}$ if and only if for each $\lambda\in\Gamma$, $d_\lambda(f(x), f(y)) \leqq d_\lambda(x, y)$, for all $x, y \in X$.

(ii) f is contractive w.r.t. $\{d_\lambda\}_{\lambda\in\Gamma}$ if and only if f is nonexpansive w.r.t. $\{d_\lambda\}_{\lambda\in\Gamma}$ and for any $x, y\in X$, if $x\neq y$, then there is a $\lambda\in\Gamma$ such that $d_\lambda(f(x), f(y)) < d_\lambda(x, y)$.