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# THE CLASS OF $(p, q)$-BIHARMONIC FUNCTIONS

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**Our main interest in this paper is with the $(p, q)$-biharmonic boundary value problem, which takes the following form: Given continuous functions $\phi$ and $\Psi$ on Wiener's or Royden's $p$-and $q$-harmonic boundaries $\alpha$ and $\beta$ respectively, find a function $u$ satisfying $(\Delta + q)(\Delta + p)u = 0$ and**

$$u \mid \alpha = \varphi , \qquad u \mid \beta = \Psi .$$

**We shall solve this problem by what we call the $(p, q)$-biharmonic projection.**

In §1 we give some preliminary results. The $(p, q)$-biharmonic projection is introduced in §2 for various classes of functions, and in §4 for suitably restricted Riemannian manifolds. In §3 we characterize classes of manifolds with respect to significant subclasses of $(p, q)$-quasiharmonic functions by means of the $p$-harmonic Green's function and the $q$-elliptic measure on $R$. The $(p, q)$-quasiharmonic nondegeneracies of the manifold are the various conditions we impose on $R$ in §4. Finally in §5 we give some explicit results concerning certain classes of density functions.

1. On a smooth noncompact Riemannian manifold $R$ of dimension $m \geqq 2$ with a smooth metric tensor $(g_{ij})$, the Laplace-Beltrami operator is given by

$$\Delta \cdot = - \frac{1}{\sqrt{g}} \sum_{i=1}^{m} \frac{\partial}{\partial x^i} \sum_{j=1}^{m} \sqrt{g}\, g^{ij} \frac{\partial \cdot}{\partial x^j} ,$$

where $x = (x^1, \cdots, x^m)$ is a local coordinate system, $g = \det (g_{ij})$, and $(g^{ij}) = (g_{ij})^{-1}$. Let $p(x)$ be a density function, that is, a nonnegative $C^2$ function on $R$. A $p$-harmonic function is a $C^2$ solution of the equation $\Delta_p u = 0$ with

$$\Delta_p = \Delta + p .$$

We call a $C^4$ function $(p, q)$-*biharmonic* if it satisfies the equation

$$\Delta_q \Delta_p u = 0 ,$$

and we denote by $W_{pq} = W_{pq}(R)$ the family of $(p, q)$-biharmonic functions on $R$. An important subclass of $W_{pq}$ is the class $Q_{pq} = Q_{pq}(R)$ of $(p, q)$-*quasiharmonic* functions, i.e., the $C^2$ solutions of $\Delta_p u = e_q$, where $e_q$ is the $q$-elliptic measure on $R$ (see No. 2).