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STRONG HEREDITY IN RADICAL CLASSES

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In a recent paper, W. G. Leavitt has called a radical class $\mathscr{P}$ in a universal class $\mathscr{W}$ of not necessarily associative rings strongly hereditary if $\mathscr{P}(I) = I \cap \mathscr{P}(R)$ for all ideals I of any ring $R \in \mathscr{W}$. In this paper, strongly hereditary radicals are investigated and a new construction is provided for the minimal strongly hereditary radical containing a given class in $\mathscr{W}$. Nonassociative versions of some results of E. P. Armendariz on semisimple classes are proved, including a characterization of semisimple classes corresponding to strongly hereditary radicals.

Unless otherwise indicated, $\mathscr{W}$ is assumed to be a universal class of not necessarily associative rings. If $\mathscr{P}$ is any radical class in $\mathscr{W}$, we denote the class of $\mathscr{P}$-semisimple rings in $\mathscr{W}$ by $\mathscr{S}\mathscr{P}$. We use the notation $I \leqq R$ to denote that I is an ideal of R. For any class $\mathscr{M}$ we denote by $\mathscr{H}\mathscr{M}$ and $\mathscr{I}\mathscr{M}$, respectively, the homomorphic closure and ideal closure of $\mathscr{M}$.

For any radical class $\mathscr{P} \subseteqq \mathscr{W}$, Leavitt in [7] has defined $\mathscr{G}\mathscr{P} = \{J' \mid J \leqq I \leqq R, J \in \mathscr{P}$, and J' is the ideal of R generated by $J\}$. Radical classes $\mathscr{P}$ for which $\mathscr{P} = \mathscr{G}\mathscr{P}$ are said to satisfy property (a). Theorem 1 of [7] states that a hereditary radical class $\mathscr{P}$ is strongly hereditary if and only if $\mathscr{P}$ satisfies property (a). In [8], it is shown that any subclass $\mathscr{M}$ of $\mathscr{W}$ is contained in a unique minimal radical class satisfying property (a).

Some preliminary results are required.

LEMMA 1.1. [2]. *Let $\mathscr{P}$ be any radical class in $\mathscr{W}$. Then $\mathscr{S}\mathscr{P}$ is hereditary if and only if for each $R \in \mathscr{W}$ with $I \leqq R$ we have* $\mathscr{P}(I) \subseteqq (R)$.

LEMMA 1.2. *Let $\mathscr{P}$ be any radical class. Then $\mathscr{P}$ is strongly hereditary if and only if both $\mathscr{P}$ and $\mathscr{S}\mathscr{P}$ are hereditary.*

Proof. If $\mathscr{P}$ is strongly hereditary, $\mathscr{P}(I) = I \cap \mathscr{P}(R)$ for each $I \leqq R$, so $\mathscr{P}$ and $\mathscr{S}\mathscr{P}$ are hereditary. Suppose $\mathscr{P}$ and $\mathscr{S}\mathscr{P}$ are hereditary and let $I \leqq R$. By Lemma 1.1 $\mathscr{P}(I) \subseteqq I \cap \mathscr{P}(R)$. Also since $\mathscr{P}(R) \in \mathscr{P}$ and $\mathscr{P}$ is hereditary, $I \cap \mathscr{P}(R) \in \mathscr{P}$. Since $I \cap \mathscr{P}(R) \leqq I$, we have $I \cap \mathscr{P}(R) \subseteqq \mathscr{P}(I)$.

LEMMA 1.3. *Let $\mathscr{P}$ be a radical class satisfying property* (a). *Then $\mathscr{S}\mathscr{P}$ is hereditary. If $\mathscr{P}$ is hereditary, $\mathscr{P}$ satisfies property*