PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 1, 1972

TOPOLOGIES ON SEQUENCE SPACES

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A study is made of two means to topologize a space of sequences. The first method rests upon the duality of every sequence space S with the sequence space φ (finitely $\neq 0$) by means of the form

$$((a_j), (b_j)) = \sum_j a_j b_j \quad (a_j) \in S, (b_j) \in \varphi .$$

The second method is a generalization of the Köthe-Toeplitz duality theory. The Köthe dual S^α of a sequence space S consists of all (b_j) such that $(a_j b_j) \in l^1$ (absolutely convergent series) for $(a_j) \in S$. Other spaces may take the role of l^1 in the above definition. A means to construct a topology on S is determined using this generalized dual. Finally, a particularly suitable type of space (the sum space) to play the role of l^1 is defined.

Our motivation is primarily the inexact but nevertheless meaningful question: what is the "natural" topology for an arbitrary space of sequences S. We consider two classes of topologies on S. Both classes include the topologies studied by Köthe and Toeplitz [10] and Garling [3, 4]. Our most important result is Theorem 4.10 which establishes a relationship between these two classes.

The first method of topologizing a space of sequences is based upon the observation that every sequence space S is in duality with the space φ of finitely nonzero sequences by means of the natural pairing

$$((a_j), (b_j)) = \sum_j a_j b_j \quad (a_j) \in S, (b_j) \in \varphi .$$

It is thus possible to define upon S topologies having a neighborhood base at 0 consisting of polars of a subfamily of the collection of all S-bounded subsets of φ. A few basis observations are made concerning this topology in § 3.

The second method is a direct extension of the Köthe-Toeplitz duality theory [10]. The Köthe-Toeplitz dual, S^α, of a sequence space S consists of all sequences (b_j) such that $\sum_{j=1}^\infty |a_j b_j| < \infty$ for each (a_j) in S. In other words, S^α consists of all (b_j) such that $(a_j b_j) \in l^1$ (absolutely convergent series) for each $(a_j) \in S$. It is easy to see how S and S^α are in duality. In § 4 we examine the consequences of allowing other spaces to play the role of l^1 in the above alternative definition. Thus for S a sequence space and T a sequence space with a linear topological structure S^T consists of all sequences (b_j) such that